*V. Ya. Arsenin*

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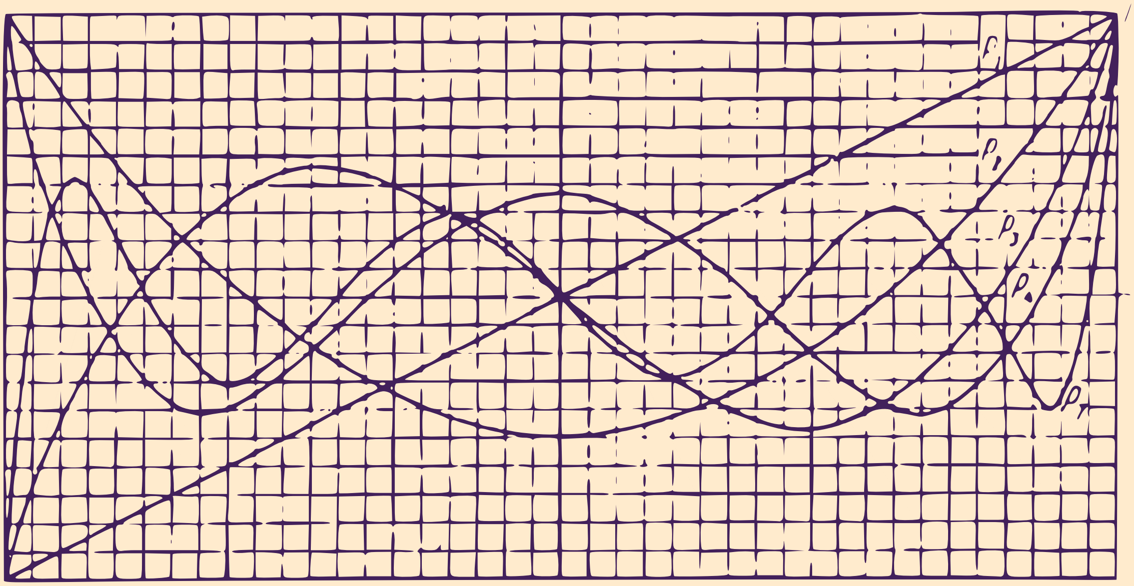

Basic Equations and Special Functions of Mathematical Physics

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# BASIC EQUATIONS AND SPECIAL FUNCTIONS OF MATHEMATICAL PHYSICS

V. Ya. Arsenin

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LONDON ILIFFE BOOKS LTD
NEW YORK AMERICAN ELSEVIER PUBLISHING
COMPANY INC.

First published by Nauka, Moscow, under the title Matematicheskaya Fizika — Osnovnye Uravneniya i Spetsial'nye Funktsii

English Edition first published in 1968 by Iliffe Books Ltd,
42 Russell Square, London, W.C.1.

American edition published by American Elsevier Publishing Company Inc.
52 Vanderbilt Avenue, New York, New York 10017

Library of Congress Catalog Card Number 68-57390

Translated and prepared for press by Scripta Technica Ltd,
139a New Bond Street, London, W.1.

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Printed in Poland

# Contents

# Preface

This book consists of two parts. Part 1 gives an account of the methods for solving typical problems in mathematical physics and an introduction to integral equations. Part 2 deals with the applications of these methods to problems requiring the use of special functions.

Extensive use is made in this book of the Dirac $\delta$-function. Generalised functions are introduced and their applications are described. Each chapter concludes with a number of problems illustrating the main text (altogether 150 problems with answers are given).

This text was designed for students of physics and engineering and was based on a course given at the Department of Theoretical and Experimental Physics of the Moscow Engineering-Physics Institute.

Special thanks are due to A. N. Tikhonov, who initiated this course, and to A. A. Samarskii for many discussions and valuable advice. V. S. Vladimirov and T. F. Volkov have read the manuscript and put forward a number of important suggestions which have been incorporated in the text. I am also indebted to the publisher's reader, S. A. Shirokova, for many useful suggestions which have led to an improvement in the presentation of the material.

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# PART ONE

# 1

# Linear Equations with Two Independent Variables

Many physical problems lead to second-order partial differential equations. These equations can be written in the general form of a functional relationship between the independent variables $x_1, \ldots, x_n$, the unknown function $u$ and its first- and second-order partial derivatives $u_{x_1}, \ldots, u_{x_n}$; $u_{x_1x_2}, \ldots, u_{x_1x_n}, \ldots, u_{x_ix_j}, \ldots, u_{x_nx_n}$:

$$\Phi(x_1, x_2, \ldots, x_n; u, u_{x_1}, u_{x_2}, \ldots, u_{x_n}; u_{x_1x_1}, \ldots, u_{x_ix_j}, \ldots, u_{x_nx_n}) = 0$$

Very frequently these equations are linear in the second-order derivatives, i.e. they are of the form

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}u_{x_ix_j} + F(x_1, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n}) = 0$$

where the coefficients of the highest-order derivatives, $a_{ij}$, are functions of only the independent variables $x_1, x_2, \ldots, x_n$.

When the function $F(x_1, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n})$ is linear in the arguments $u, u_{x_1}, \ldots, u_{x_n}$, the differential equation is termed 'linear'. Linear equations are of the form

$$\sum_{i=1}^{n}\sum_{j=1} a_{ij}u_{x_ix_j} + \sum_{i=1} b_iu_{x_i} + cu = f(x_1, \ldots, x_n) \qquad (1)$$

where the coefficients $a_{ij}$, $b_i$, $c$ are functions only of the independent variables $x_1, \ldots, x_n$.

When $f(x_1, \ldots, x_n) \equiv 0$, Equation (1) is called a homogeneous

linear equation; and when this identity is not satisfied, it is called an inhomogeneous equation.

When the coefficients $a_{ij}$, $b_i$, $c$ are constants, Equation (1) is called a linear equation with constant coefficients.

Equations which are linear in the second-order derivatives or in both second- and first-order derivatives can be divided into three classes (types). Each class includes an equation with a particularly simple form which is known as the canonical form. Equations belonging to a given class have many common properties. To investigate these properties it is sufficient to consider the canonical equations. In the next few chapters we shall be concerned with the properties of the solutions of canonical equations and with methods for obtaining these solutions.

Whether or not a particular equation belongs to a given class is determined by the coefficients of the highest-order derivatives. We shall consider this classification first for equations in which the unknown function $u$ depends on only two variables, i.e. $u = u(x, y)$. In this case, equations which are linear in the highest-order derivatives may be written in the form

$$a_{11}u_{xx}+2a_{12}u_{xy}+a_{22}u_{yy}+F(x, y, u, u_x, u_y) = 0 \tag{1a}$$

Similarly, linear equations may be written in the form

$$a_{11}u_{xx}+2a_{12}u_{xy}+a_{22}u_{yy}+b_1u_x+b_2u_y+cu = f(x, y) \tag{2}$$

where $a_{ij}$, $b_i$, $c$ are functions of only the independent variables $x$, $y$. Any equations of the form (1a) or (2) can be reduced to a more simple form, the canonical form, by a suitable transformation of the independent variables. In studying partial differential equations with two independent variables, we shall therefore be able to confine our attention to the canonical equations.

Let us transform the independent variables in Equation (1a) in accordance with the formulae

$$\xi = \varphi(x, y), \qquad \eta = \psi(x, y) \tag{3}$$

which define a one-to-one mapping between the points $(\xi, \eta)$ and $(x, y)$. We shall require that the functions $\varphi(x, y)$ and $\psi(x, y)$ and their first- and second-order derivatives be continuous. We then have

$$u_x = \varphi_x u_\xi+\psi_x u_\eta, \qquad u_y = \varphi_y u_\xi+\psi_y u_\eta$$

$$u_{xx} = \varphi_x^2 u_{\xi\xi}+2\varphi_x\psi_x u_{\xi\eta}+\psi_x^2 u_{\eta\eta}+\varphi_{xx}u_\xi+\psi_{xx}u_\eta$$

$$u_{yy} = \varphi_y^2 u_{\xi\xi}+2\varphi_y\psi_y u_{\xi\eta}+\psi_y^2 u_{\eta\eta}+\varphi_{yy}u_\xi+\psi_{yy}u_\eta$$

$$u_{xy} = \varphi_x\varphi_y u_{\xi\xi}+(\varphi_x\psi_y+\varphi_y\psi_x)u_{\xi\eta}+\psi_x\psi_y u_{\eta\eta}+\varphi_{xy}u_\xi+\psi_{xy}u_\eta$$

Substituting these derivatives into (1a) and collecting up terms involving the same derivatives, we obtain the transformed equation

$$\alpha_{11}u_{\xi\xi}+2\alpha_{12}u_{\xi\eta}+\alpha_{22}u_{\eta\eta}+F_1(u_\xi, u_\eta, u, \xi, \eta) = 0 \tag{4}$$

where

$$\begin{aligned}
\alpha_{11} &= a_{11}\varphi_x^2+2a_{12}\varphi_x\varphi_y+a_{22}\varphi_y^2\\
\alpha_{12} &= a_{11}\varphi_x\psi_x+a_{12}(\varphi_x\psi_y+\varphi_y\psi_x)+a_{22}\varphi_y\psi_y\\
\alpha_{22} &= a_{11}\psi_x^2+2a_{12}\psi_x\psi_y+a_{22}\psi_y^2
\end{aligned} \tag{5}$$

Using (5) we can readily verify that

$$\alpha_{12}^2-\alpha_{11}\alpha_{22} \equiv (a_{12}^2-a_{11}a_{22})\left[\frac{D(\varphi;\psi)}{D(x;y)}\right]^2 \tag{6}$$

We can now adopt the following classification of equations of the form (1a). If in a given domain $D$ the discriminant $\nabla^2 = a_{12}^2-a_{11}a_{22}$ is positive ($\nabla^2>0$) then, Equation (1a) is hyperbolic in $D$. If, on the other hand, $\nabla^2<0$ in $D$, (1a) is elliptic in $D$. Finally, when $\nabla^2 \equiv 0$ at all points of $D$, (1a) is parabolic in $D$.

It follows from the identity given by (6) that the type of an equation of the form (1a) is unaffected by a transformation of the independent variables of the form (3). We shall use this procedure of transformation of the independent variables to simplify (1a), i.e. to reduce it to its canonical form. For each type of equation there is a particular canonical form.

1. If (1a) is hyperbolic in a domain $D$, there exist in $D$ functions $\varphi(x, y)$ and $\psi(x, y)$ such that the transformation given by (3) reduces (1a) to the simpler form

$$u_{\xi\eta}+F_1(u_\xi, u_\eta, u, \xi, \eta) = 0 \tag{7}$$

This is the canonical form.

We shall now summarise the procedure for finding $\varphi(x, y)$ and $\psi(x, y)$ without discussing the conditions for their existence:

a. If $a_{11} = a_{22} = 0$ in $D$, then $a_{12} \neq 0$. Dividing both sides of (1) by $2a_{12}$ we obtain the canonical form (7).

b. Suppose that $a_{11}^2+a_{22}^2 \neq 0$ in $D$. We shall confine our attention to the case when at least one of the coefficients $a_{11}$, $a_{22}$ is not identically zero in a part $D_1$ of $D$. Suppose $a_{11}$ has this property.

We shall take $\varphi(x, y)$ and $\psi(x, y)$ in (3) to be functions which ensure that the coefficients $\alpha_{11}$ and $\alpha_{22}$ in the transformed equation (4)

will vanish, i.e. we shall suppose that they are the solutions of the following equations:

$$
\begin{aligned}
a_{11}\varphi_x^2+2a_{12}\varphi_x\varphi_y+a_{22}\varphi_y^2 &= 0\\
a_{11}\psi_x^2+2a_{12}\psi_x\psi_y+a_{22}\psi_y^2 &= 0
\end{aligned}
\tag{8}
$$

These yield

$$
\frac{\varphi_x}{\varphi_y} = \frac{-a_{12}\pm\sqrt{\nabla^2}}{a_{11}}, \qquad \frac{\psi_x}{\psi_y} = \frac{-a_{12}\pm\sqrt{\nabla^2}}{a_{11}}
$$

Consequently, each of the equations in (8) can be split into the following:

$$
\varphi_x+\lambda_1(x, y)\varphi_y = 0, \qquad \psi_x+\lambda_2(x, y)\psi_y = 0 \tag{9}
$$

where

$$
\lambda_1(x, y) = \frac{a_{12}-\sqrt{\nabla^2}}{a_{11}}, \qquad \lambda_2(x, y) = \frac{a_{12}+\sqrt{\nabla^2}}{a_{11}} \tag{10}
$$

The two equations in (9) are equivalent to

$$
\frac{dy}{dx} = \lambda_1(x, y), \qquad \frac{dy}{dx} = \lambda_2(x, y) \tag{11}
$$

This procedure yields functions $\varphi(x, y)$ and $\psi(x, y)$ which will ensure that the coefficients $\alpha_{11}$ and $\alpha_{22}$ will vanish in the case under consideration. At the same time, $\alpha_{12} \neq 0$, which follows immediately from the identity given by (6). Dividing the transformed equation by $2\alpha_{12}$ we obtain the required canonical form.

On integrating (11) we obtain

$$
\varphi(x, y) = c_1 \quad \text{and} \quad \psi(x, y) = c_2
$$

which form two families of curves called the characteristic curves, or simply characteristics, of (1a). We note that two characteristics belonging to different families will never coincide since $\lambda_1 \neq \lambda_2$. It follows that the above families of characteristic curves form curvilinear coordinate nets.

2. If (1a) is elliptic in $D$, there exist in $D$ functions $\varphi(x, y)$ and $\varphi(x, y)$ such that (1a) can be reduced to the canonical form

$$
u_{\xi\xi}+u_{\eta\eta}+F_1(u_\xi, u_\eta, u, \xi, \eta) = 0 \tag{12}
$$

by means of the transformation (3). We shall again confine our attention simply to a description of the procedure for finding $\varphi(x, y)$ and $\psi(x, y)$.

As before, we first reduce the equation formally to the form

$$u_{\xi\eta}+F_1(u_\xi, u_\eta, u, \xi, \eta) = 0 \tag{13}$$

The new variables $\xi$ and $\eta$ will now be complex conjugates

$$\xi = \varphi(x, y)+i\psi(x, y), \qquad \eta = \varphi(x, y)-i\psi(x, y)$$

since the differential equations for the characteristic curves are now of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a_{12}}{a_{11}}-i\frac{\sqrt{-\nabla^2}}{a_{11}}, \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a_{12}}{a_{11}}+i\frac{\sqrt{-\nabla^2}}{a_{11}}$$

Consequently, an elliptic equation has only imaginary characteristic curves.

Finally we can transform the independent variables again in accordance with the formulae

$$\rho = \frac{\xi+\eta}{2} = \varphi(x, y), \qquad \sigma = \frac{\xi-\eta}{2i} = \psi(x, y)$$

so that (13) and, consequently, (1a) will assume the required canonical form (in the new variables)

$$u_{\rho\rho}+u_{\sigma\sigma}+F_2(u_\rho, u_\sigma, u, \rho, \sigma) = 0$$

3. If (1a) is parabolic in $D$, there exist in $D$ functions $\varphi(x, y)$ and $\psi(x, y)$ such that the transformation given by (3) reduces Equation (1a) to the canonical form

$$u_{\eta\eta}+F_1(u_\xi, u_\eta, u, \xi, \eta) = 0 \tag{14}$$

The procedure for finding the functions $\varphi(x, y)$ and $\psi(x, y)$ can be summarised as follows. To begin with, we must find the function $\varphi(x, y)$ which will ensure that the coefficient $\alpha_{11}$ in the transformed equation will vanish, i.e. it will be a solution of the equation

$$a_{11}\varphi_x^2+2a_{12}\varphi_x\varphi_y+a_{22}\varphi_y^2 = 0 \tag{15}$$

As in the case of hyperbolic equations we are assuming that $a_{11}$ is not identically zero in any part $D_1$ of $D$. We then solve (17) for $\varphi_x/\varphi_y$, but in contrast to the hyperbolic case (see (9)), we obtain only one equation, namely,

$$\varphi_x+\lambda(x, y)\varphi_y = 0 \tag{16}$$

where

$$\lambda(x, y) = \frac{a_{12}}{a_{11}}$$

Any solution of (16) which is not identically equal to a constant can be taken for $\varphi(x, y)$. The coefficient $\alpha_{12}$ in the transformed equation will then also vanish, which follows from the condition that (1a) is a parabolic equation, and from the identity given by (6). For $\psi(x, y)$ we can take any twice continuously differentiable function which does not lead to $\alpha_{22}$ being zero. Dividing the equation transformed in this way by $\alpha_{22}$, we obtain the required canonical form. An equation of the parabolic type has only the single family of characteristic curves

$$\frac{dy}{dx} = \lambda(x, y)$$

If the initial equation (1a) is linear, the transformed equation will clearly also be linear.

Thus, the canonical forms of linear equations are

$$\begin{aligned} u_{\xi\eta}+\beta_1 u_\xi+\beta_2 u_\eta+\gamma u &= f(\xi, \eta) \text{ (hyperbolic)} \\ u_{\xi\xi}+u_{\eta\eta}+\beta_1 u_\xi+\beta_2 u_\eta+\gamma u &= f(\xi, \eta) \text{ (elliptic)} \\ u_{\eta\eta}+\beta_1 u_\xi+\beta_2 u_\eta+\gamma u &= f(\xi, \eta) \text{ (parabolic)} \end{aligned} \tag{17}$$

4. If the differential equation is a linear equation with constant coefficients, then the coefficients $\beta_1, \beta_2, \gamma$ in the corresponding canonical equations will also be constants. (The characteristic curves of a hyperbolic equation will then be straight lines.) The equations given by (17) can then be simplified still further by substituting

$$u = v\, e^{\mu\xi+\nu\eta} \tag{18}$$

where $\mu$, $\nu$ are to be determined.

Evaluating the derivatives of the function $u$ and substituting the result into, say, the first equation in (17), we obtain

$$v_{\xi\eta}+(\nu+\beta_1)v_\xi+(\mu+\beta_2)v_\eta+(\mu\nu+\mu\beta_1+\nu\beta_2+\gamma)v = f(\xi, \eta)\, e^{-\mu\xi-\nu\eta}$$

If we substitute

$$\mu = -\beta_2, \qquad \nu = -\beta_1$$

the transformed equation assumes the form

$$v_{\xi\eta}+\gamma_1 v = f_1(\xi, \eta) \tag{19}$$

where

$$\gamma_1 = \gamma-\beta_1\beta_2, \qquad f_1(\xi, \eta) = f(\xi, \eta)\, e^{\beta_2\xi+\beta_1\eta}$$

Similarly, an elliptic equation will reduce to the form

$$v_{\xi\xi}+v_{\eta\eta}+\gamma_1 v = f_1(\xi, \eta) \tag{20}$$

where

$$\gamma_1 = \gamma-\frac{1}{4}(\beta_1^2+\beta_2^2), \quad \mu = -\frac{1}{2}\beta_1, \quad \nu = -\frac{1}{2}\beta_2, \quad f_1 = f\mathrm{e}^{-\mu\xi-\nu\xi}$$

In a parabolic equation, the coefficients of $v_\xi$ and $v_\eta$ cannot be made to vanish by a suitable choice of $\mu$ and $\nu$ since the transformed equation is of the form

$$v_{\eta\eta}+\beta_1 v_\xi+(2\nu+\beta_2)v_\eta+(\nu^2+\nu\beta_2+\mu\beta_1+\gamma)v = f_1(\xi, \eta)$$

Substituting $\nu = -\frac{1}{2}\beta_2$, $\mu = \frac{1}{\beta_1}\left(\frac{1}{4}\beta_2^2-\gamma\right)$ we obtain

$$v_{\eta\eta}+\beta_1 v_\xi = f_1(\xi, \eta) \tag{21}$$

As a result of the simplifications of (1a) described above, we will confine ourselves to consideration of methods for solving problems which can be formulated in the form of canonical equations. In the case of linear equations with constant coefficients, we may generally confine our attention to equations of the form given by (19), (20) and (21).

Let us now consider a number of examples.

*Example 1* $u_{xx}-yu_{yy} = 0$. In this case, $a_{11} = 1$, $a_{12} = 0$, $a_{22} = -y$, $\nabla^2 = a_{12}^2-a_{11}a_{22} = y$. Consequently, for $y > 0$ the equation is of the hyperbolic type, whereas for $y < 0$ it is of the elliptic type.

a. Consider, to begin with, the region in which the equation is hyperbolic. The differential equations for the characteristic curves are of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\sqrt{y}, \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{y}$$

and their general integrals are $x-2\sqrt{y} = c_1$, $x+2\sqrt{y} = c_2$. Substituting

$$\xi = x-2\sqrt{y}, \qquad \eta = x+2\sqrt{y}$$

we obtain the canonical form of the transformed equation

$$u_{\xi\eta}+\frac{1}{2}\frac{1}{\xi-\eta}(u_\xi-u_\eta) = 0$$

The characteristics are the right- and left-hand branches of the family of parabolae $(x-c)^2 = 4y$ (solid and broken curves in Fig. 1.1). The minimum points of the parabolae, which lie on the $x$ axis, do not belong to the characteristic curves since, at these points, the equation is not of the hyperbolic type ($\nabla^2 = 0$).

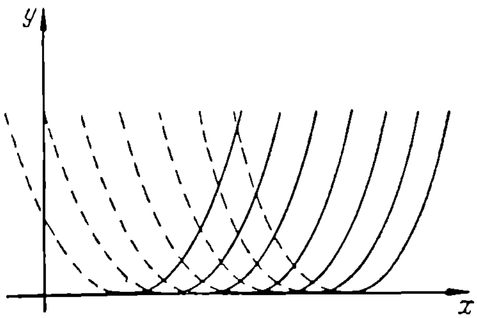



Fig. 1.1

b. In the region in which the equation is elliptic ($y < 0$), we shall substitute

$$\rho = \frac{\xi+\eta}{2} = x, \qquad \sigma = \frac{\eta-\xi}{2i} = 2\sqrt{-y}$$

The canonical form of the equation is

$$u_{\rho\rho}+u_{\sigma\sigma}-\frac{1}{\sigma}u_\sigma = 0$$

*Example 2* $\quad xu_{xx}-2\sqrt{xy}\,u_{xy}+yu_{yy}+\frac{1}{2}u_y = 0.$
In this case, $a_{11} = x$, $a_{12} = -\sqrt{xy}$, $a_{22} = y$, $\nabla^2 = a_{12}^2-a_{11}a_{22} \equiv 0.$ Consequently, this equation is hyperbolic at all points. It has the single family of characteristic curves given by the differential equation

$$\frac{dy}{dx} = -\sqrt{\frac{y}{x}} \qquad \left(\text{or } \frac{dy}{\sqrt{y}} = \frac{-dx}{\sqrt{x}}\right)$$

The general integral of this equation is

$$\sqrt{x}+\sqrt{y} = c$$

We shall therefore substitute

$$\xi = \sqrt{x}+\sqrt{y}$$

and $\eta$ can be set equal to any function $\varphi(x, y)$ which does not lead to $\alpha_{22}$ being zero in the transformed equation. Let

$$\eta = \sqrt{x}$$

The canonical form of the equation is then

$$u_{\eta\eta}-\frac{1}{\eta}(u_\xi+u_\eta)=0$$

5. Whether or not a linear differential equation which does not contain a mixed derivative of the unknown function, i.e. an equation of the form

$$a_{11}u_{xx}+a_{22}u_{yy}+b_1u_x+b_2u_y+cu=f(x, y) \tag{22}$$

belongs to a particular class, is determined by the signs of $a_{11}$ and $a_{22}$. More precisely, if $a_{11}(x, y)$ and $a_{22}(x, y)$ have different signs throughout $D$ (and do not vanish in $D$), (22) is hyperbolic in $D$. If, on the other hand, $a_{11}(x, y)$ and $a_{22}(x, y)$ have the same signs throughout $D$ (and do not vanish in $D$), Equation (22) is elliptic in $D$. Finally, if one of the coefficients $a_{11}$, $a_{22}$ is zero throughout $D$, (22) is parabolic in $D$.

Similar criteria can be used to classify linear equations of the form

$$\sum_{i=1}^{n} a_{ii}u_{x_i x_i}+\sum_{k=1}^{n} b_k u_{x_k}+cu=f(x_1, x_2, \ldots, x_n) \tag{23}$$

with a large number of independent variables $(x_1, x_2, \ldots, x_n)$, where $a_{ij}, b_k, c$ are functions of the variables $(x_1, x_2, \ldots, x_n)$. Equation (23) can be classified as follows. It is:

elliptic at the point $(x_1^0, x_2^0, \ldots, x_n^0)$ if all the coefficients $a_{ii}(x_1^0, x_2^0, \ldots, x_n^0)$ at this point are not zero and have the same sign;

hyperbolic at the point $(x_1^0, x_2^0, \ldots, x_n^0)$ if the coefficients $a_{ii}(x_1^0, x_2^0, \ldots, x_n^0)$ at this point are not all zero and all of them, except one (for example $a_{i_0 i_0}$), have the same sign;

hyperbolic at the point $(x_1^0, x_2^0, \ldots, x_n^0)$ if the coefficients $a_{ii}(x_1^0, x_2^0, \ldots, x_n^0)$ at this point are all, except one (for example, $a_{i_0 i_0}$), different from zero and all have the same sign

$$a_{i_0 i_0}(x_1^0, x_2^0, \ldots, x_n^0)=0 \quad \text{and} \quad b_{i_0}(x_1^0, x_2^0, \ldots, x_n^0)\neq 0$$

PROBLEMS

1. Reduce the following equations to canonical form:

(a) $x^2u_{xx}-y^2u_{yy}=0$

(b) $y^2u_{xx}+x^2u_{yy}=0$

(c) $x^2u_{xx}+2xyu_{xy}+y^2u_{yy}=0$

(d) $u_{xx}+yu_{yy}+0.5\,u_y=0$

2. Reduce the following equations to canonical form:

(a) $u_{xx}+2u_{xy}+u_{yy}+3u_x-5u_y+4u=0$

(b) $u_{xx}+4u_{xy}+3u_{yy}+5u_x+u_y+4u=0$

(c) $2u_{xx}+2u_{xy}+u_{yy}+4u_x+4u_y+u=0$

# 2

# Partial Differential Equations in Physics Boundary-value Problems

## 2.1 SMALL TRANSVERSE VIBRATIONS OF A STRING

We shall assume that the string is elastic and offers no resistance to any change of form other than a change in length. This is expressed mathematically by saying that the tension in the string is always parallel to the tangent to the instantaneous profile of the

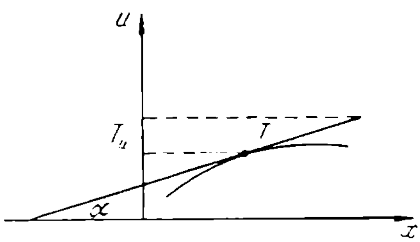



*Fig. 2.1*

string (Fig. 2.1). We shall also assume that the vibrations are small, i.e. the square of the displacement can be neglected, and that they take place in the $(x, u)$ plane.

The assumption that the vibrations are small ensures that the tension $T$ in the string is independent of time $t$. To show this, consider a segment $(x_1, x_2)$ of the undisturbed string. Its initial length is $x_2 - x_1$, whereas the length at time $t$ is

$$\int_{x_1}^{x_2} \sqrt{1+u_x^2}\, dx \approx \int_{x_1}^{x_2} 1 \cdot dx = x_2 - x_1$$

It follows that, to within terms of the second order in $u_x$, the length

of a fixed segment of the string is independent of time, i.e. the segment is not extended. Hence, in view of Hooke's law, it follows that the tension $T$ is also independent of time (to within terms of the second order in $u_x$). Consequently, $T$ can be a function of only $x$:

$$T = T(x)$$

Since we are considering the transverse vibrations of the string, we shall only be interested in the component of the tension along the $u$ axis, which will be denoted by $T_u$. It is clear that

$$T_u = T\sin\alpha = T\tan\alpha\cos\alpha = T\frac{u_x}{\sqrt{1+u_x^2}} \approx Tu_x$$

where $\alpha$ is the angle between the tangent to the curve $u = u(x, t)$ and the $x$ axis for given $t$ (Fig. 2.1). The momentum of a segment $(x_1, x_2)$ of the string at time $t$ is equal to

$$\int_{x_1}^{x_2} u_t(\xi, t)\rho\,(\xi)\,\mathrm{d}\xi$$

where $\rho$ is the linear density of the string.

According to Newton's second law, the change in the momentum of the segment $(x_1, x_2)$ in a time $\nabla^2 t = t_2 - t_1$ is equal to the time integral of the applied forces which, in the present case, consist of the tension force $Tu_x$ at the ends of the segment and the external force $\int_{x_1}^{x_2} f(\xi, t)\,\mathrm{d}\xi$, expressed in terms of a force density $f(x, t)$:

$$\int_{x_1}^{x_2} [u_t(\xi, t_2) - u_t(\xi, t_1)]\,\rho(\xi)\,\mathrm{d}\xi$$

$$= \int_{t_1}^{t_2} [T(x_2)u_x(x_2, \tau) - T(x_1)u_x(x_1, \tau)]\,\mathrm{d}\tau + \int_{t_1}^{t_2}\int_{x_1}^{x_2} f(\xi, \tau)\,\mathrm{d}\xi\,\mathrm{d}\tau \qquad (1)$$

This is the integral form of the equation for small transverse vibrations of a segment of a string.

If $u(x, t)$ has continuous second-order derivatives and $T(x)$ has a continuous first-order derivative, then using the mean value theorem for the integrals in (1) we obtain

$$u_{tt}(\xi_1, \tau_1)\rho(\xi_1)\nabla^2 t\,\nabla^2 x = \frac{\partial}{\partial x}[T(x)u_x]_{\substack{x=\xi_2\\ t=\tau_2}}\nabla^2 t\,\nabla^2 x + f(\xi_3, \tau_3)\nabla^2 t\,\nabla^2 x \quad (2)$$

where $\xi_1, \xi_2, \xi_3 \in [x_1, x_2]$, $\tau_1, \tau_2, \tau_3 \in [t_1, t_2]$. Dividing both sides of (2) by $\nabla^2 t\, \nabla^2 x$ and proceeding to the limit as $\nabla^2 t \to 0$ and $\nabla^2 x \to 0$, we obtain the following differential equation for the small transverse vibrations of a string

$$\frac{\partial}{\partial x}[Tu_x]+f(x, t) = \rho(x)u_{tt} \tag{3}$$

When $T = \text{const}$ and $\rho = \text{const}$, this equation is usually written in the form

$$a^2 u_{xx}+F(x, t) = u_{tt} \tag{4}$$

where $a^2 = T/\rho$, $F(x, t) = (1/\rho)f(x, t)$. Equation (4) is called the one-dimensional wave equation.

## 2.2 SMALL LONGITUDINAL VIBRATIONS OF AN ELASTIC ROD

Consider a rod parallel to the $x$ axis. We shall use the following notation: $S(x)$ is the cross-sectional area of the rod in the plane perpendicular to the $x$ axis at the point $x$, $k(x)$ and $\rho(x)$ is the Young's modulus and density of the rod in a plane having the abscissa $x$, and $u(x, t)$ is the displacement (in the direction of the axis of the rod) of the section having the abscissa $x$ at time $t$. We shall assume that the displacement of all points on the plane with given $x$ is the same. It is clear that the longitudinal vibrations are completely described by the function $u(x, t)$. Small longitudinal vibrations will be defined as those for which the stresses produced in the rod during the vibration always obey Hooke's law. Consider the relative extension of a segment $(x, x+\nabla^2 x)$ at time $t$. The coordinates of the end of this segment are

$$x+u(x, t), \qquad x+\nabla^2 x+u(x+\nabla^2 x, t)$$

Consequently, the relative change in the length of the segment is

$$\frac{\{[x+\nabla^2 x+u(x+\nabla^2 x, t)]-[x+u(x, t)]\}-\nabla^2 x}{\nabla^2 x} = u_x(x+\theta\nabla^2 x, t)$$

$$(0 < \theta < 1)$$

It follows that the relative extension at the point $x$ at time $t$ is $u_x(x, t)$ and the tension $T$ is, in view of Hooke's law, given by

$$T = k(x)S(x)u_x(x, t)$$

If we now apply Newton's second law to the segment $(x_1, x_2)$ of the rod in a time interval $\nabla^2 t = t_2 - t_1$, we obtain

$$\int_{x_1}^{x_2} \{u_t(\xi, t_2) - u_t(\xi, t_1)\}\rho(\xi) S(\xi)\, d\xi$$

$$= \int_{t_1}^{t_2} \{S(x_2)k(x_2)\, u_x(x_2, \tau) - S(x_1)k(x_1)\, u_x(x_1, \tau)\}\, d\tau + \int_{t_1}^{t_2}\int_{x_1}^{x_2} f(\xi, \tau)\, d\xi\, d\tau$$

where $f(x, t)$ is the density of the external force acting on a cross-section at a distance $x$ from the origin. This is the integral form of the equation for the small longitudinal vibrations of the rod. Assuming that the function $u(x, t)$ has continuous second-order derivatives and that $k(x)$ and $S(x)$ have continuous first-order derivatives, we can readily show that the differential equation for the small longitudinal vibrations of a rod is

$$\frac{\partial}{\partial x}[S(x)k(x)u_x(x, t)] + f(x, t) = \rho(x)S(x)u_{tt}(x, t) \tag{5}$$

When $S(x)$, $k(x)$ and $\rho(x)$ are constants, Equation (5) assumes the form

$$a^2 u_{xx} + F(x, t) = u_{tt}$$

where

$$a^2 = \frac{k}{\rho}, \qquad F(x, t) = \frac{1}{\rho S} f(x, t)$$

Equations (3) and (5) are identical in form and differ only in the notation ($Sk$ instead of $T$, and $\rho S$ instead of $\rho$). They are hyperbolic at all points since, by definition, $T(x)$, $S(x)$ and $k(x)$ are positive.

## 2.3 SMALL TRANSVERSE VIBRATIONS OF A MEMBRANE

A membrane is defined as a stretched plane sheet which is perfectly flexible and can be displaced, but offers resistance to stretching. For example, a plate whose thickness is small in comparison with its two other linear dimensions can in certain cases be regarded as a membrane. We shall consider small transverse vibrations of a membrane at right-angles to its own plane $(x, y)$ and will assume that the squares of the displacements $u_x$ and $u_y$ can be neglected, where $u = u(x, y, t)$ is the displacement of the point $(x, y)$ at time $t$.

Let $ds$ be an element of arc of a contour on the surface of the membrane and let $N$ be a point on this element. The tension force acting on this element is $\boldsymbol{T}$ $ds$. The fact that the membrane is perfectly flexible is expressed mathematically by saying that the vector $\boldsymbol{T}$ lies in the plane which is tangential to the surface of the membrane at the point $M$ and perpendicular to the element $ds$, and that the magnitude of $\boldsymbol{T}$ at this point is independent of the direction of $ds$. Since the vibrations are small, it follows that the component $T_c$ of the tension vector $\boldsymbol{T}$ on the $(x, y)$ plane is equal to $T$. In fact, $T_c = T\cos\alpha$, where $\alpha$ is the angle between $\boldsymbol{T}$ and the $(x, y)$ plane. However, $\alpha$ is not greater than the angle $\gamma$ between the tangent plane to the surface of the membrane, which contains the vector $\boldsymbol{T}$, and the $(x, y)$ plane (i.e. $\alpha \leqslant \gamma$). It follows that

$$\cos\alpha \geqslant \cos\gamma = \frac{1}{\sqrt{1+u_x^2+u_y^2}} \approx 1$$

Consequently, $\cos\alpha \approx 1$ and hence $T_c \approx T$.

The second consequence of the fact that the vibrations are small is that the tension $T$ is independent of the time $t$. To show this, consider a part $S$ of the undisturbed membrane. This area is equal to $\iint_S dx\,dy$. At time $t$ this area is equal to

$$\iint_S \frac{dx\,dy}{\cos\gamma} \approx \iint_S dx\,dy$$

It follows that the area of a fixed part of the membrane is independent of time, and in view of Hooke's law, $T$ is independent of time. Since $\boldsymbol{T}$ is perpendicular to the element of arc $ds$, it follows that $T$ is also

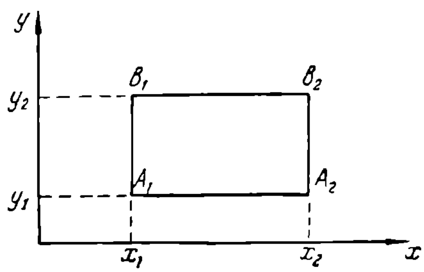


*Fig. 2.2*

independent of $x$ and $y$. In fact, consider a part $A_1B_1B_2A_2$ of the undisturbed membrane bounded by lines parallel to the coordinate axes (Fig. 2.2):

the tension force acting on the segment of the surface is equal to

$$\int_{A_1A_2} T\,\mathrm{d}s + \int_{A_2B_2} T\,\mathrm{d}s + \int_{B_2B_1} T\,\mathrm{d}s + \int_{B_1A_1} T\,\mathrm{d}s$$

Since points of the membrane are not displaced in the direction of the $x$ and $y$ axes, we have

$$\int_{A_2B_2} T\,\mathrm{d}s + \int_{B_1A_1} T\,\mathrm{d}s = \int_{y_1}^{y_2} T(x_2, y)\,\mathrm{d}y - \int_{y_1}^{y_2} T(x_1, y)\,\mathrm{d}y$$

$$= \int_{y_1}^{y_2} [T(x_2, y) - T(x_1, y)]\,\mathrm{d}y = 0 \tag{6}$$

and

$$\int_{A_1A_2} T\,\mathrm{d}s + \int_{B_2B_1} T\,\mathrm{d}s = \int_{x_1}^{x_2} [T(x, y_1) - T(x, y_2)]\,\mathrm{d}x = 0 \tag{7}$$

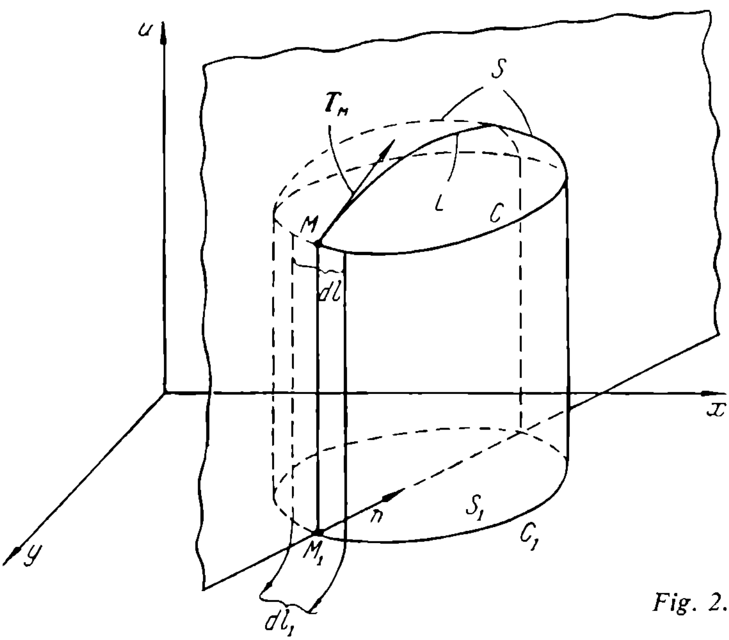



Fig. 2.3

Since the intervals $(x_1, x_2)$ and $(y_1, y_2)$ are arbitrary, it follows from (6) and (7) that $T(x_1, y) = T(x_2, y)$ and $T(x, y_1) = T(x, y_2)$, and so on.

Let $S$ be a segment of the membrane at time $t$, bounded by the contour $C$ (Fig. 2.3). Let $S_1$ and $C_1$ be the projections of $S$ and $C$ on to the $(x, y)$ plane.

To determine the vertical component $P_u$ of the tension force acting on $C$, consider an element $\mathrm{d}l$ of $C$ and a point $M$ on it. Let $T_M$ be the tension vector at the point $M$ at right-angles to $\mathrm{d}l$. The plane drawn through $T_M$ which is perpendicular to the $(x, y)$ plane

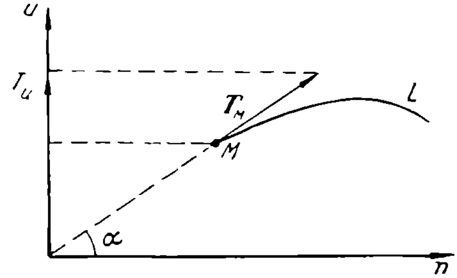



*Fig. 2.4*

will cut the $(x, y)$ plane along the normal $n$ to $C_1$ at the point $M_1$ (Fig. 2.3). Fig. 2.4 shows the curve $L$ along which this plane cuts the surface $S$. It is clear that

$$T_u = T\sin\alpha = T\frac{\tan\alpha}{\sqrt{1+\tan^2\alpha}} = T\frac{\dfrac{\partial u}{\partial n}}{\sqrt{1+\left(\dfrac{\partial u}{\partial n}\right)^2}} \approx T\frac{\partial u}{\partial n}$$

Consequently,

$$P_u = \int\limits_C T_u\,\mathrm{d}l = \int\limits_C T\frac{\partial u}{\partial n}\,\mathrm{d}l = \int\limits_{C_1} T\frac{\partial u}{\partial n}\frac{\mathrm{d}l_1}{\cos\beta}$$

where $\beta$ is the angle between the elements $\mathrm{d}l$ and $\mathrm{d}l_1$.

Since $\beta \leqslant \gamma$ (as in the condition $\alpha \leqslant \gamma$ discussed above) $\cos\beta \geqslant \cos\gamma = 1/\sqrt{1+u_x^2+u_y^2} \approx 1$. Therefore,

$$P_u = T\int\limits_{C_1} \frac{\partial u}{\partial n}\,\mathrm{d}l_1$$

Application of Gauss's theorem to this integral yields

$$P_u = \iint\limits_{S_1} (u_{xx}+u_{yy})\,\mathrm{d}x\,\mathrm{d}y = T\iint\limits_{S_1} \nabla^2 u\,\mathrm{d}x\,\mathrm{d}y$$

The equation for the small transverse vibrations of a membrane can now readily be obtained. Let $f(x, y, t)$ be the density of the resultant external force acting on the membrane at the point $M(x, y)$ at time $t$ along the $u$ axis, and let $\rho(x, y)$ be the surface density of the membrane. Applying Newton's second law to a part $S_1$ of the

membrane in a time interval $\nabla^2 t = t_2 - t_1$, we obtain the required integral equation

$$\iint_{S_1} [u_t(x, y, t_2) - u_t(x, y, t_1)]\rho(x, y)\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_{t_1}^{t_2}\iint_{S_1} T\,\nabla^2 u\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau + \int_{t_1}^{t_2}\iint_{S_1} f(x, y, \tau)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau$$

Assuming the existence and continuity of the corresponding derivatives, we readily find the following differential equation for the small transverse vibration of a membrane

$$T\,\nabla^2 u + f(x, y, t) = \rho u_{tt}$$

This is a hyperbolic equation. When $\rho$ = const, it can be rewritten in the form

$$a^2\,\nabla^2 u + F(x, y, t) = u_{tt} \tag{8}$$

where $a^2 = T/\rho$, $F(x, y, t) = (1/\rho)\,f(x, y, t)$. Equation (8) is called the two-dimensional wave equation.

## 2.4 THE EQUATIONS OF HYDRODYNAMICS AND ACOUSTICS

The motion of a continuous medium can be characterised by a velocity vector $\boldsymbol{v}(x, y, z, t)$, a pressure $p(x, y, z, t)$ and a density $\rho(x, y, z, t)$. We will take the medium to be an ideal liquid (gas). Consider a volume $D$ of the liquid bounded by the surface $S$. The pressure acting on this volume is equal to

$$\iint_S p\boldsymbol{n}\,\mathrm{d}s$$

where $\boldsymbol{n}$ is a unit vector in the direction of the inward normal to $S$. Gauss's theorem yields

$$\iint_S p\boldsymbol{n}\,\mathrm{d}s = -\iiint_D \nabla p\,\mathrm{d}\tau$$

where $\nabla p$ is the pressure gradient. In the absence of external forces, the equation of motion for the volume $D$ can be written in the form

$$\iiint_D \rho\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\,\mathrm{d}\tau = -\iiint_D \nabla p\,\mathrm{d}\tau$$

and, since $D$ is arbitrary, this yields the following equation of motion in Euler's form

$$\rho\frac{d\boldsymbol{v}}{dt}+\nabla p = 0 \tag{9}$$

where $d\boldsymbol{v}/dt$ is the acceleration of a particle and is equal to

$$\frac{\partial \boldsymbol{v}}{\partial t}+v_1\frac{\partial \boldsymbol{v}}{\partial x}+v_2\frac{\partial \boldsymbol{v}}{\partial y}+v_3\frac{\partial \boldsymbol{v}}{\partial z}$$

If there are no sources or sinks within $D$, the time rate of change of the amount of liquid in $D$ is equal to the flux of the liquid through $S$, i.e.

$$\frac{\partial}{\partial t}\iiint\limits_{D}\rho\, d\tau = -\iint\limits_{S}\rho\,(\boldsymbol{v}\cdot\boldsymbol{n})\,ds$$

Using Gauss's theorem on the right-hand side of this equation, we obtain

$$\iiint\limits_{D}\left[\frac{\partial\rho}{\partial t}+\operatorname{div}(\rho\boldsymbol{v})\right]d\tau = 0$$

which yields the continuity equation for the medium

$$\frac{\partial\rho}{\partial t}+\operatorname{div}(\rho\boldsymbol{v}) = 0 \tag{10}$$

Consider the adiabatic motion of a gas for which

$$p = p_0\left(\frac{\rho}{\rho_0}\right)^{\gamma} \tag{11}$$

where $\gamma = c_p/c_v$, $c_p$ and $c_v$ are the specific heats at constant pressure and constant volume, respectively, and $p_0, \rho_0$ are the initial values of the pressure and density, respectively. The non-linear equations given by (9)–(11) form a complete system of equations describing the adiabatic motion of an ideal gas. If we substitute

$$\sigma = \frac{\rho-\rho_0}{\rho_0}, \qquad \rho = \rho_0(1+\sigma) \tag{12}$$

and confine our attention to small vibrations for which we can neglect second order terms involving the ratio $\sigma$, the velocity, and the gradients of velocity and pressure, then equations (9) and (11)

can be considerably simplified (this is referred to as linearisation). In fact, subject to these assumptions, we have

$$\frac{1}{\rho} = \frac{1}{\rho_0}\frac{1}{1+\sigma} = \frac{1}{\rho_0}(1-\sigma+\sigma^2-\ldots) \approx \frac{1}{\rho_0}(1-\sigma)$$

$$p = p_0(1+\sigma)^\gamma \approx p_0(1+\gamma\sigma) \tag{13}$$

$$\frac{1}{\rho}\nabla p \approx \frac{1}{\rho_0}(1-\sigma)\nabla p \approx \frac{\gamma p_0}{\rho_0}\nabla\sigma \qquad (p_0 = \text{const})$$

$$\operatorname{div}(\rho \boldsymbol{v}) \approx \rho_0 \operatorname{div}[(1+\sigma)\boldsymbol{v}] \approx \rho_0 \operatorname{div}(\boldsymbol{v}) \qquad (\rho_0 = \text{const})$$

Neglecting the higher-order terms in (9) and (10) we obtain

$$\begin{aligned} &\boldsymbol{v}_t + a^2\nabla\sigma = 0 \qquad \left(a^2 = \frac{\gamma p_0}{\rho_0}\right) \\ &\sigma_t + \operatorname{div}(\boldsymbol{v}) = 0 \end{aligned} \tag{14}$$

Let us apply the divergence operator to the first equation in (14) and the operator $\partial/\partial t$ to the second. Subtracting one from the other, we obtain

$$a^2\nabla^2\sigma = \sigma_{tt} \tag{15}$$

From (12), (13) and (14) we obtain the analogous equations for $\rho$ and $p$

$$\begin{aligned} a^2\nabla^2\rho &= \rho_{tt} \\ a^2\nabla^2 p &= p_{tt} \end{aligned} \tag{16}$$

Equations (15) and (16) are the equations of acoustics. They are of the hyperbolic type. Such equations are also referred to as the three-dimensional wave equations.

From the first equation in (14) we find that

$$\boldsymbol{v}(x, y, z, t) = \boldsymbol{v}(x, y, z, 0) - a^2\int_0^t \nabla\sigma \, d\tau$$

$$= \boldsymbol{v}(x, y, z, 0) - \nabla\left(\int_0^t a^2\sigma \, d\tau\right)$$

If we assume that at the initial time ($t = 0$) the velocity field has a potential $f(x, y, z)$, i.e.

$$\boldsymbol{v}|_{t=0} = -\nabla f(x, y, z)$$

then $\boldsymbol{v}(x, y, z, t) = -\nabla\left\{f(x, y, z) + a^2\int_0^t \sigma \, d\tau\right\} = -\nabla u$

Consequently, the velocity field has a potential $u$ when $t > 0$ and this is given by

$$u = f(x, y, z) + a^2 \int_0^t \sigma \, d\tau$$

Differentiating this with respect to time, we find that

$$u_t = a^2 \sigma$$

$$u_{tt} = a^2 \sigma_t$$

Substituting for $\sigma_t$ and $\boldsymbol{v}$ in terms of $u$ into the second equation in (14), we obtain

$$a^2 \nabla^2 u = u_{tt} \tag{17}$$

It follows that the potential of the velocity field is also a solution of the wave equation.

## 2.5 ELECTRIC FIELD IN A VACUUM

Maxwell's equations in a vacuum may be written in the form

$$\begin{aligned} \operatorname{curl} \boldsymbol{E} &= -\frac{1}{c}\frac{\partial \boldsymbol{H}}{\partial t} \\ \operatorname{div} \boldsymbol{E} &= 0 \\ \operatorname{div} \boldsymbol{H} &= 0 \\ \operatorname{curl} \boldsymbol{H} &= \frac{1}{c}\frac{\partial E}{\partial t} \end{aligned} \tag{18}$$

where $\boldsymbol{H}$ is the magnetic field and $\boldsymbol{E}$ is the electric field. Application of the curl operator to the first equation yields

$$\operatorname{curl} \operatorname{curl} \boldsymbol{E} = -\frac{1}{c}\frac{\partial}{\partial t} \operatorname{curl} \boldsymbol{H} \tag{19}$$

Moreover,

$$\operatorname{curl} \operatorname{curl} \boldsymbol{E} = \nabla(\operatorname{div} \boldsymbol{E}) - \nabla^2 \boldsymbol{E}$$

In our case, curl curl $\boldsymbol{E} = -\nabla^2\boldsymbol{E}$, since div $\boldsymbol{E} \equiv 0$. Substituting this into (19) and using the last equation in (18), we obtain the wave equation for $\boldsymbol{E}$:

$$c^2\nabla^2\boldsymbol{E} = \boldsymbol{E}_{tt} \tag{20}$$

## 2.6 THE EQUATIONS OF HEAT TRANSFER AND DIFFUSION

We shall now derive the equation describing the distribution of temperature in a body. Let $u(M, t)$ be the temperature of the body at the point $M$ at time $t$. In deriving the required equation we shall use Fourier's law for the heat flux density $w$ in the direction $\boldsymbol{n}$ per unit time

$$w = -k\frac{\partial u}{\partial n}$$

where $k$ is the thermal conductivity. The thermal conductivity may be a function of temperature, position and time

$$k = k(u, M, t)$$

Consider a part $D$ of the body bounded by the surface $S$. Let $f(M, t)$ be the density of heat sources and let us evaluate the heat balance for $D$ in a small time interval $\nabla^2 t$:

$$Q_1 = \iiint\limits_D f(M, t)\,\mathrm{d}\tau\,\nabla^2 t$$

$$Q_2 = -\iint\limits_S k\frac{\partial u}{\partial n}\,\mathrm{d}\tau\,\nabla^2 t$$

where $Q_1$ represents the heat gain from the sources and $Q_2$ the heat lost from $D$. The derivative $\partial u/\partial n$ is evaluated in the direction of the outward normal to the surface $S$. Finally,

$$Q_3 = \iiint\limits_D c\rho u_t\,\mathrm{d}\tau\,\nabla^2 t$$

is the heat lost due to change in temperature, where $c$ is the specific heat and $\rho$ the density of the material. Since energy must be conserved, we have

$$Q_1 = Q_2 + Q_3$$

or

$$\iint\limits_S k\frac{\partial u}{\partial n}\,\mathrm{d}\sigma + \iiint\limits_D f(M, t)\,\mathrm{d}\tau = \iiint\limits_D c\rho u_t\,\mathrm{d}\tau$$

Application of Green's theorem to the first integral yields

$$\iiint\limits_D [\operatorname{div}(k\,\nabla u)+f(M,\,t)]\,d\tau = \iiint\limits_D c\rho u_t\,d\tau$$

and since $D$ is arbitrary, we obtain the required heat transfer equation

$$\operatorname{div}(k\,\nabla u)+f(M,\,t) = c\rho u_t \tag{21}$$

The diffusion equation can be established in a similar way. According to Nernst's law for the flux of matter $w$ in the direction $n$

$$w = -D\frac{\partial u}{\partial n}$$

where $u = u(M, t)$ is the concentration of the diffusing material (gas or liquid) and $D$ is the diffusion coefficient. In the equation for $Q_3$ we must now replace $c\rho$ by the porosity $c$ of the medium in which the diffusion takes place. The diffusion equation is found to be of the form

$$\operatorname{div}(D\,\nabla u)+f(M,\,t) = cu \tag{22}$$

Physical considerations indicate that the coefficients $k$ and $D$ must be positive, and, therefore, Equations (21) and (22) are of the parabolic type.

Problems involving the determination of the steady-state temperature or concentration lead to the elliptical equation

$$\operatorname{div}(k\,\nabla u) = -f(M) \tag{23}$$

if $k$, $c$, $\rho$ and $f$ (and, correspondingly, $D$ and $c$) are independent of time $t$.

## 2.7 FORMULATION OF BOUNDARY-VALUE PROBLEMS

The solution of many problems in physics and other branches of science by mathematical methods requires a preliminary mathematical formulation of these problems. This means that one must first write down the equation or system of equations which the required function or system of functions describing the phenomenon under investigation must satisfy. The next step is to write down the additional conditions which the unknown function must fulfil on the boundaries of the region within which it is to be determined. The

nature of these additional conditions can be illustrated by considering the problems considered in the last sections. For example, in the case of the vibrations of a string or rod (Equations (3) and (5)), one must prescribe the initial profile

$$u(x, 0) = \varphi(x)$$

and the initial velocity

$$u_t(x, 0) = \psi(x)$$

of all points on the string (rod). These are the initial conditions. There are analogous conditions for any other wave equation. Moreover, we must write down the conditions at the ends of the string (rod). Thus, if the law of motion of the ends ($x = 0$ and $x = l$) is specified to be

$$u(0, t) = \mu_1(t), \qquad u(l, t) = \mu_2(t)$$

these additional conditions involving the value of the variable (displacement) on the boundary will be called boundary conditions of type I. If we specify the variation of the force applied to the end of a string (rod) and acting in the direction of the vibrations, the conditions at the ends can be written in the form

$$Eu_x|_{x=0} = f_1(t), \qquad Eu_x|_{x=l} = f_2(t)$$

or

$$u_x(0, t) = \nu_1(t), \qquad u_x(l, t) = \nu_2(t)$$

These involve the derivative of the variable on the boundary and are called boundary conditions of type II.

Suppose that a spring is attached to the end of the rod at $x = l$ and acts in the direction of the $x$ axis. The tension force $Eu_x$ at the end will then be balanced by the force $\alpha u$ due to the spring. The boundary conditions at the end can then be written in the form

$$Eu_x(l, t) = -\alpha u(l, t)$$

where $\alpha$ is the spring constant, or

$$u_x(l, t) + hu(l, t) = 0$$

If the spring, in its turn, moves in accordance with the law $x = \beta(t)$, the boundary conditions can be represented by

$$u_x(l, t) + h[u(l, t) - \beta(t)] = 0$$

This is called a boundary condition of type III. At the left-hand end ($x = 0$) it can be written in the form

$$u_x(0, t) - h[u(0, t) - \beta(t)] = 0$$

For the two- and three-dimensional cases the above three types of boundary condition are of the form

$$u|_S = \mu(M, t) \qquad \text{(type I)} \tag{24}$$

$$\left.\frac{\partial u}{\partial n}\right|_S = \nu(M, t) \qquad \text{(type II)} \tag{25}$$

$$\left.\left(\frac{\partial u}{\partial n} + hu\right)\right|_S = \beta(M, t) \qquad \text{(type III)} \tag{26}$$

where $\partial u/\partial n$ is the derivative along the outward normal to the surface $S$.

Similar boundary conditions are encountered in problems leading to parabolic equations. Thus, if the temperature on the surface of a body is specified, we have a boundary condition of type I. If we specify the heat flux $k\,\partial u/\partial n$ through the surface $S$ of the body we have a boundary condition of type II. If, finally, the heat transfer between the surface of the body and the surrounding medium at temperature $\beta(M, t)$ is in accordance with Newton's law

$$-k\frac{\partial u}{\partial n} = h[u-\beta]\Big|_S$$

we have the boundary condition of type III.

Other types of boundary condition will be discussed later. The boundary conditions considered above are linear because the unknown function or its derivatives enter into them in linear forms. They are called homogeneous if the right-hand sides $(\mu, \nu, \beta)$ are identically zero, and inhomogeneous in all other cases. It is evident that similar boundary conditions are encountered in problems leading to elliptic equations. Physical interpretation of each presents no special difficulties.

We shall now formulate the corresponding three types of boundary condition for equations of the form

$$\operatorname{div}(k\nabla u) - qu + f(M, t) = \rho u_{tt} \tag{27}$$

and

$$\operatorname{div}(k\nabla u) - qu + f(M, t) = \rho u_t \tag{28}$$

where $k$, $q$ and $\rho$ are functions of the coordinates of $M$. All the equations considered above belong to this type with $q \equiv 0$.

*First boundary-value problem* Find the function $u(M, t)$ satisfying Equation (27) (or (28)) in the domain $(M \in D, t > 0)$ and the additional conditions.

1. The initial conditions

$$u(M,0)=\varphi(M), \qquad u_t(M,0)=\psi(M), \qquad M\in D$$

(or $u(M,0)=\varphi(M)$).

2. The boundary conditions

$$u(M,t)|_S=\mu(M,t), \qquad t>0$$

*Second and third boundary-value problems* These can be formulated in a similar way by replacing the boundary condition of type I by a boundary condition of type II or III.

*Note* All the above types of boundary condition can be represented by the single relation

$$\left\{\gamma_1(M)\frac{\partial u}{\partial n}+\gamma_2(M)u\right\}_S=\beta(M,t)$$

When $\gamma_1\equiv 0$ we obtain a boundary condition of type I, $\gamma_2\equiv 0$ yields a boundary condition of type II and $\gamma_1\neq 0$, $\gamma_2\neq 0$ yield a boundary condition of type III.

It is easy to imagine problems in which one is interested in the unknown function $u(M,t)$ at points $M$ which are so far from the boundary $S$ that the effect of the boundary conditions on these points can be neglected. This justifies the following formulation.

*Cauchy's problem* Find the function $u(M,t)$ which satisfies Equation (27) (or (28)) for $t>0$ at any point $M$ of space, and the initial conditions

$$u(M,0)=\varphi(M), \qquad u_t(M,0)=\psi(M)$$

(or $u(M,0)=\varphi(M)$).

If we are interested in the values of the function $u(M,t)$ for large $t$, the effect of the initial conditions can be neglected. This leads to the following problem.

*Determination of steady-state conditions* It is required to find the function $u(M,t)$ which satisfies Equation (27) (or (28)) in $D$ and the boundary condition

$$\left(\gamma_1\frac{\partial u}{\partial n}+\gamma_2 u\right)_S=\beta(M,t)$$

(or no initial conditions).

For an elliptic equation, the boundary-value problems can be formulated as follows. It is required to find a function $u(M)$ which satisfies in the domain $D$ the equation

$$\operatorname{div}[k(M)\nabla u] - q(M)u = -f(M)$$

and the boundary condition

$$\left\{\gamma_1(M)\frac{\partial u}{\partial n} + \gamma_2(M)u\right\}_S = \beta(M)$$

on the boundary $S$. $\gamma_1 \equiv 0, \gamma_2 \equiv 0$ and $\gamma_1 \neq 0, \gamma_2 \neq 0$ correspond to the first, second and third boundary-value problems, respectively.

## PROBLEMS

1. The upper end of an elastic, uniform, vertically suspended, heavy rod of length $l$ is rigidly attached to the ceiling of a freely falling lift which, having reached a velocity $v_0$, is instantaneously brought to rest. Formulate the boundary-value problem for the longitudinal vibrations of the rod.

2. Formulate the boundary-value problem for the small transverse vibrations of a string in a medium with a resistance proportional to velocity, assuming that the ends of the string are rigidly fixed.

3. An elastic rod ($0 \leqslant x \leqslant l$) of variable cross-section $S(x)$ has its ends fixed elastically, using springs. Formulate the boundary-value problem for the longitudinal vibrations of the rod, neglecting the deformation of the transverse cross-sections.

4. Formulate the boundary-value problem for the transverse vibrations of a heavy string relative to the vertical equilibrium position, if its upper end ($x = 0$) is rigidly fixed and the lower end is free.

5. Consider the previous problem on the assumption that the string rotates with an angular velocity $\omega = \text{const}$ about the vertical equilibrium position.

6. A light string rotating around the vertical axis with a constant angular velocity $\omega$ is in the horizontal plane and one of its ends ($x = 0$) is attached to a point on the axis, whilst the other end is free. At the initial time $t = 0$ all the points on the string are given small deflections and velocities at right-angles to this plane. Formulate the boundary-value problem for the deflections of points on the string from the plane of equilibrium motion.

7. An elastic cylinder is displaced from its equilibrium state at time $t = 0$ in such a way that its transverse cross-sections are rotated through small angles $\theta$ within their own planes about the axis of the cylinder. Formulate the boundary-value problem for the resulting small torsional vibrations of the cylinder if its ends are rigidly fixed (or are free).

8. A current $I(t)$ is passed at time $t = 0$ through a string $(0 \leqslant x \leqslant l)$, whose ends are rigidly fixed and whose resistanceis negligible. The string is in a magnetic field $H$. Formulate the boundary-value problem for the transverse vibrations of the string under the action of the ponderomotive forces.

9. Two semi-infinite, elastic rods with identical transverse cross-sections are joined end to end and form a single infinite rod. If the densities and elastic moduli of the two rods are $\rho_1$, $E_1$ and $\rho_2$, $E_2$, formulate the boundary problem for the small longitudinal vibrations of the rod under the action of an initial perturbation.

10. Formulate the boundary-value problem for the transverse vibrations of a string with fixed ends, carrying a localised mass $m$ at the point $x_0$.

11. Consider the transverse vibrations of an infinite string under the action of a force $F(t)$ applied at time $t = 0$ at the point $x = 0$ and moving along the string with velocity $v_0$.

12. Find the equations for the a.c. voltage and current (i.e. the equations of telegraphy) in a thin conductor with resistance $R$, capacitance $C$, self-inductance $L$ and leakage loss $G$ per unit length. Hint: use Ohm's law and the conservation of charge.

13. Formulate the boundary-value problem for the electrical oscillations in a conductor with negligible resistance and loss (but finite inductance and capacitance per unit length) if the ends of the conductor are earthed, one through a lumped resistance $R_0$ and the other through a lumped capacitance $C_0$.

14. Consider the previous problem on the assumption that one of the ends of the conductor $(x = 0)$ is earthed through a lumped self-inductance $L_0^{(1)}$ and an e.m.f. $E(t)$ is applied to the other end through a lumped self-inductance $L_0^{(2)}$.

15. Formulate the problem of the electrical oscillations in a loss-less infinite conductor consisting of two semi-infinite conductors connected through a lumped capacitance $C_0$.

16. The lateral surface of a thin rod surrounded by a medium at a temperature $u_m = \varphi(t)$ loses heat in accordance with Newton's law of cooling. Formulate the boundary-value problem for the

temperature distribution in the rod if one end of it is maintained at the temperature $f_1(t)$ and a heat flux $q(t)$ is applied to the other.

17. Formulate the boundary-value problem for the temperature distribution in a rod carrying a constant electric current $I$ when the surface of the rod loses heat to the surrounding medium maintained at zero temperature according to Newton's law, and the ends of the rod are held in large clamps of given thermal capacity and high thermal conductivity.

18. Derive the diffusion equation for a medium moving with velocity $v(x)$ in the direction of the $x$ axis if the surfaces of equal concentration at each instant of time are planes perpendicular to the $x$ axis.

19. Derive the diffusion equation for a stationary medium whose particles (a) decay (for example, an unstable gas) at a rate proportional to the concentration, and (b) multiply (for example, neutrons) at a rate proportional to their concentration.

20. Formulate the problem for the determination of the temperature distribution in an infinite rod produced by joining two semi-infinite rods of different materials if the two rods are joined (a) directly and (b) through a massive clamp of thermal capacity $C_0$ and very high thermal conductivity.

21. Formulate the boundary-value problem for a semi-infinite rod, one end of which burns in such a way that the combustion front propagates at velocity $v$ and has a temperature $\varphi(t)$.

22. Formulate the problem of the determination of the heating of an infinitely thin rod exposed to a point source of heat of strength $Q$ moving along it with a velocity $v_0$. Assume that all the heat is communicated to the rod and that the thermal capacity of the source is negligible.

23. Formulate the boundary-value problem for the cooling of a thin circular ring, assuming Newton's law of cooling and that the temperature of the surrounding medium is $u_0$.

24. Derive the equation for the propagation of a plane electromagnetic field in a conducting medium (i.e. in a medium in which the displacement currents can be neglected in comparison with the conduction currents).

25. Use Maxwell's equations to derive the equation for the electrostatic potential produced by a charged conductor, and for the potential in a current-carrying conductor.

# 3

# The Method of Characteristics

In this chapter we shall be mainly concerned with the simplest wave equation, i.e.

$$a^2 u_{xx} + f(x, t) = u_{tt} \tag{1}$$

The method of characteristics can be used to obtain solutions for a number of problems which can be formulated in terms of this equation. The principle of this method is best understood in terms of an example involving the solution of Cauchy's problem (Section 2.7), for the homogeneous wave equation.

## 3.1 VIBRATIONS OF AN INFINITE STRING D'ALEMBERT'S FORMULA

**3.1.1** It is required to find the function $u(x, t)$ which is continuous in the closed region $(-\infty < x < \infty, t \geqslant 0)$ and satisfies the equation

$$a^2 u_{xx} = u_{tt} \qquad (-\infty < x < \infty, t > 0) \tag{2}$$

and the initial conditions

$$u(x, 0) = \varphi(x), \qquad u_t(x, 0) = \psi(x) \qquad (-\infty < x < +\infty) \tag{3}$$

To solve this problem we shall reduce Equation (2) in terms of its characteristic curves. In the present case

$$a_{11} = a^2, \qquad a_{12} = 0, \qquad a_{22} = -1, \qquad \nabla^2 = a^2$$

Consequently, the differential equations for the characteristic curves are

$$\frac{dt}{dx} = \frac{1}{a}, \qquad \frac{dt}{dx} = -\frac{1}{a}$$

and $x-at = c_1$, $x+at = c_2$ are their general integrals. Let us substitute

$$\xi = x-at, \qquad \eta = x+at$$

The transformed equation is now of the form

$$u_{\xi\eta} = 0 \tag{4}$$

If we assume that the required solution exists, then by substituting it into (2) we obtain an identity. Consequently, the transformed Equation (4) will also be an identity. Integrating this identity with respect to $\eta$ we obtain

$$u_\xi = \Phi_1(\xi) \tag{5}$$

where $\Phi_1(\xi)$ is an arbitrary function. Integrating the identity given by (5) with respect to $\xi$, we obtain

$$u = \int \Phi_1(\xi)\,d\xi + F(\eta) = \Phi(\xi) + F(\eta)$$

or

$$u(x, t) = \Phi(x-at) + F(x+at) \tag{6}$$

where $\Phi(\xi)$ and $F(\eta)$ are arbitrary functions.

Thus, by assuming the existence of the solution of Cauchy's problem, we have come to the conclusion that it should be of the form given by (6). To ensure that the functions $u(x, t)$ given by (6) are, in fact, the solutions of (2), it is necessary that the functions $\Phi(z)$ and $F(z)$ have first- and second-order derivatives. Subject to these conditions, direct verification will show that each of the functions $\Phi(x-at)$ and $F(x+at)$ is a solution of (2).

Solutions of the form given by (6) include those which satisfy the prescribed initial conditions (3):

$$u(x, 0) = \varphi(x) \equiv \Phi(x) + F(x)$$
$$u_t(x, 0) = \psi(x) \equiv -a\Phi'(x) + aF'(x)$$

Integrating the last identity we obtain two equations for $\Phi(z)$ and $F(z)$

$$\begin{cases} \Phi(y) + F(y) \equiv \varphi(y) \\ -\Phi(y) + F(y) \equiv \dfrac{1}{a} \displaystyle\int_{x_0}^{y} \psi(z)\,dz + C \end{cases} \tag{7}$$

from which we find that

$$F(y) = \frac{\varphi(y)}{2} + \frac{1}{2a}\int_{x_0}^{y} \psi(z)\,\mathrm{d}z + \frac{C}{2}$$

$$\Phi(y) = \frac{\varphi(y)}{2} - \frac{1}{2a}\int_{x_0}^{y} \psi(z)\,\mathrm{d}z - \frac{C}{2}$$

Substituting these functions into (6), we obtain d'Alembert's formula

$$u(x, t) = \frac{\varphi(x-at)+\varphi(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \psi(z)\,\mathrm{d}z \tag{8}$$

Thus, by assuming the existence of the solution of Cauchy's problem, we have come to the conclusion that it should be of the form given by (8). Consequently, it is the only solution. If the function $\varphi(x)$ possesses first- and second-order derivatives whilst $\varphi(x)$ possesses first-order derivatives, then (8) yields the required solution of Cauchy's problem, (2)–(3). This can be verified by direct substitution of the right-hand side of (8) into (2) and (3). By constructing an explicit solution of Cauchy's problem we have demonstrated its existence.

**3.1.2** Let us consider now the physical interpretation of the solution $u = \Phi(x-at)$. The function $u(x, t)$ will be called the displacement at the point $x$ at time $t$. Consider a point $x_0$ and imagine

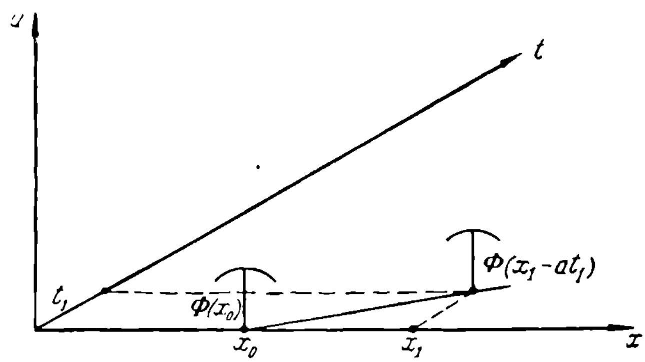



*Fig. 3.1*

that an observer begins to move with a velocity $a$ in the positive $x$ direction at time $t = 0$. At time $t_1$ it will be at the point $x_1 = x_0+at_1$. The displacement which the observer will see at the point $x_1$ at time $t_1$ will be $u = \Phi(x_1-at_1) = \Phi(x_0)$. It follows that at any given time

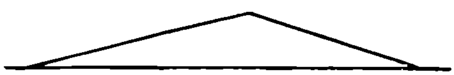
the observer will see a constant displacement $\Phi(x_0)$ at the point at which he is located. It follows that the initial profile $u(x, 0) = \Phi(x)$ will move with a velocity $a$ in the positive $x$ direction as if it were a rigid system which does not undergo a change of form (Fig. 3.1).

In view of these properties, the solution $u = \Phi(x-at)$ is called the forward travelling wave solution. A similar interpretation can be given to the solution $u = F(x+at)$. This solution is called the reverse travelling wave solution. In this case, the profile moves as if it were a rigid system moving in the negative $x$ direction with velocity $a$. It follows that any solution of (2) can be written as a superposition of forward and reverse travelling waves. The above method of obtaining the solutions of Cauchy's problem is called the method of characteristics or the method of travelling waves.

## 3.2 CONTINUOUS DEPENDENCE OF THE SOLUTION OF CAUCHY'S PROBLEM ON THE INITIAL CONDITIONS. GENERALISED SOLUTION

**3.2.1** D'Alembert's formula (8) gives the solution of Cauchy's problem (2)–(3) on the assumption that the initial functions $\varphi(x)$ and $\psi(x)$ possess bounded derivatives $\varphi'(x)$, $\varphi''(x)$, $\psi'(x)$. However, it is not difficult to find problems for which the initial functions $\varphi(x)$ and $\psi(x)$ do not have these properties. It is sufficient, for example, to specify the initial deflection of a string in the form of the broken line shown in Fig. 3.2.

*Fig. 3.2*



To show how the solution of Cauchy's problem can be obtained in such cases, we shall prove the following theorem.

*Theorem* Let $u_1(x, t)$ and $u_2(x, t)$ be the solutions of Cauchy's problem (2)–(3) subject to the initial conditions

$$u_1(x, 0) = \varphi_1(x), \qquad u_{1t}(x, 0) = \psi_1(x)$$

and

$$u_2(x, 0) = \varphi_2(x), \qquad u_{2t}(x, 0) = \psi_2(x)$$

For any $\varepsilon > 0$ and $t_1 > 0$ we can then find a quantity $\sigma > 0$, a function of $\varepsilon$ and $t_1$, which is such that the inequalities

$$|\varphi_1(x)-\varphi_2(x)| < \delta, \qquad |\psi_1(x)-\psi_2(x)| < \delta, \qquad -\infty < x < \infty$$

lead to

$$|u_1(x, t)-u_2(x, t)| < \varepsilon, \qquad -\infty < x < \infty, \qquad t \leqslant t_1$$

*Proof* Using d'Alembert's formula for $u_1(x, t)$ and $u_2(x, t)$ we obtain

$$\begin{aligned} u_1(x, t)-u_2(x, t) &= \frac{1}{2}[\varphi_1(x-at)-\varphi_2(x-at)] \\ &+\frac{1}{2}[\varphi_1(x+at)-\varphi_2(x+at)] \\ &+\frac{1}{2a}\int_{x-at}^{x+at}[\psi_1(z)-\psi_2(z)]\,\mathrm{d}z \end{aligned}$$

and, consequently,

$$\begin{aligned} |u_1(x, t)-u_2(x, t)| &\leqslant \frac{1}{2}|\varphi_1(x-at)-\varphi_2(x-at)| \\ &+\frac{1}{2}|\varphi_1(x+at)-\varphi_2(x+at)| \\ &+\frac{1}{2a}\int_{x-at}^{x+at}|\psi_1(z)-\psi_2(z)|\,\mathrm{d}z \\ &< \frac{1}{2}\delta+\frac{1}{2}\delta+\frac{1}{2a}\int_{x-at}^{x+at}\delta\,\mathrm{d}z \\ &= \delta+\delta t \leqslant \delta(1+t_1) \end{aligned}$$

If we let $\delta = \varepsilon/(1+t_1)$, the inequality

$$|u_1(x, t)-u_2(x, t)| < \varepsilon$$

will be satisfied for all $\infty < x < \infty$, $t \leqslant t_1$, which was to be proved.

The above theorem can readily be expressed in words: small changes in the initial values of Cauchy's problem lead to small changes in the solution.

In practice, the initial conditions are deduced from measurements and are therefore subject to uncertainties. The above theorem shows that small errors in the determination of the initial conditions

will lead to small changes in the solution of Cauchy's problem. The theorem also shows one of the possible ways of obtaining solutions of Cauchy's problem when the initial functions $\varphi(x)$ and $\psi(x)$ do not have the required derivative (Fig. 3.2).

**3.2.2** Let us return now to Cauchy's problem (2)–(3). We shall assume that the initial functions $\varphi(x)$ and $\psi(x)$ are not identically zero on finite segments, are continuous everywhere, and that the function $\varphi(x)$ possesses a first-order derivative. These functions can be uniformly approximated to by the differentiable functions $\varphi_n(x)$ and $\psi_n(x)$ so that

$$\varphi_n(x) \rightrightarrows \varphi(x) \qquad (n \longrightarrow \infty), \qquad \psi_n(x) \rightrightarrows \psi(x) \qquad (n \longrightarrow \infty)$$

where $\varphi_n(x)$ possesses a first and second derivative and $\psi_n(x)$ possesses a first derivative.

If we take $\varphi_n(x)$ and $\psi_n(x)$ as the initial functions for the Cauchy problem, they will define a unique solution for the problem, $u_n(x, t)$.

Consider the difference $u_{n+k}(x, t)-u_n(x, t)$. Since the sequences $\{\varphi_n(x)\}$ and $\{\psi_n(x)\}$ are uniformly convergent for any $\varepsilon > 0$ $t_1 > 0$, it is possible to find $N$ such that for any $n > N$ and any positive integral values of $k$ we have

$$|\, \varphi_n(x)-\varphi_{n+k}(x)\,| < \frac{\varepsilon}{1+t_1} \quad \text{and} \quad |\, \psi_{n+k}(x)-\psi_n(x)\,| < \frac{\varepsilon}{1+t_1}$$

for all $-\infty < x < \infty$. In view of the theorem proved above, we have the following inequalities for all $t \leqslant t_1$ and $-\infty < x < \infty$

$$|\, u_{n+k}(x, t)-u_n(x, t)\,| < \varepsilon$$

These hold for any $n > N$ and any positive integral $k$. However, this means that the sequence of solutions $\{u_n(x, t)\}$ converges uniformly in the above range of $x$, $t$ to some function $u(x, t)$. This function is called the generalised solution of Cauchy's problem (2)–(3). It is given by

$$u(x, t) = \lim_{n\to\infty} u_n(x, t) = \frac{1}{2}\lim_{n\to\infty} [\varphi_n(x-at)+\varphi_n(x+at)]$$
$$+\frac{1}{2a}\lim_{n\to\infty}\int_{x-at}^{x+at} \psi_n(z)\,dz$$

or

$$u(x, t) = \frac{\varphi(x-at)+\varphi(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \psi(z)\,dz$$

This function and its derivative $u_t(x, t)$ will assume the prescribed values $\varphi(x)$ and $\psi(x)$. It follows that in the above case, the d'Alembert formula will also yield the (generalised) solution of Cauchy's problem. The problem can also be solved in another way by using the generalised functions and their convolution (see Appendix, Section A.1).

*Definition* The fundamental solution $G(x, t)$ of the wave equation $a^2u_{xx} = u_{tt}$ is defined as the solution of the Cauchy problem

$$a^2G_{xx} = G_{tt}, \quad G(x, 0) = 0, \quad G_t(x, 0) = \delta(x) \quad .$$

It is readily verified by direct substitution that

$$G(x, t) = \frac{1}{2a}[\eta(x+at)-\eta(x-at)]$$

where $\eta(z) = \begin{cases} 1, & z > 0 \\ 0, & z < 0 \end{cases}$ is a unit step function. In point of fact (see Appendix, Section A.1),

$$G_x(x, t) = \frac{1}{2a}[\delta(x+at)-\delta(x-at)]$$

$$G_{xx}(x, t) = \frac{1}{2a}[\delta'(x+at)-\delta'(x-at)]$$

$$G_t(x, t) = \frac{1}{2}[\delta(x+at)+\delta(x-at)]$$

$$G_{tt}(x, t) = \frac{a}{2}[\delta'(x+at)-\delta'(x-at)]$$

Consequently,

$$a^2G_{xx}(x, t) \equiv G_{tt}(x, t)$$

$$G(x, 0) = \frac{1}{2a}[\eta(x)-\eta(x)] = 0$$

$$G_t(x, 0) = \frac{1}{2}[\delta(x)+\delta(x)] = \delta(x)$$

The solution of Cauchy's problem

$$a^2 v_{xx} = v_{tt}, \quad v(x, 0) = 0, \quad v_t(x, 0) = \psi(x)$$

will be written in the form of the convolution

$$v(x, t) = G(x, t) * \psi(x) \tag{9}$$

In fact, by evaluating the derivatives of the convolution (see Appendix, Section A.1), we obtain

$$v_{xx} = G_{xx} * \psi, \qquad v_{tt} = G_{tt} * \psi$$

and, consequently,

$$a^2 v_{xx} - v_{tt} \equiv (a^2 G_{xx} - G_{tt}) * \psi \equiv 0 * \psi \equiv 0$$

$$v(x, 0) = G(x, 0) * \psi(x) = 0 * \psi(x) = 0$$

$$v_t(x, 0) = G_t(x, 0) * \psi(x) = \delta(x) * \psi(x) = \psi(x)$$

The convolution $G * \psi$ can also be written in the form

$$v(x, t) = \int_{-\infty}^{\infty} G(\xi, t)\psi(x-\xi)\,\mathrm{d}\xi = \frac{1}{2a}\int_{-at}^{at} \psi(x-\xi)\,\mathrm{d}\xi$$

Substituting $x-\xi = z$ in the last integral, we obtain

$$v(x, t) = \frac{1}{2a}\int_{x-at}^{x+at} \psi(z)\,\mathrm{d}z \tag{10}$$

We therefore note that in Equation (9) and, consequently, in Equation (10) the function $\psi(x)$ can be any integrable (and even any generalised) function.

If $R(x, t)$ is a solution of Cauchy's problem,

$$a^2 R_{xx} = R_{tt}, \qquad R(x, 0) = 0, \qquad R_t(x, 0) = \varphi(x)$$

the function $w(x, t) = \dfrac{\partial}{\partial t} R(x, t)$ is a solution of the Cauchy problem

$$a^2 w_{xx} = w_{tt}, \qquad w(x, 0) = \varphi(x), \qquad w_t(x, 0) = 0$$

In fact, differentiating the identity

$$a^2 R_{xx} \equiv R_{tt}$$

with respect to $t$, we obtain $a^2(R_t)_{xx} \equiv (R_t)_{tt}$, i.e.

$$a^2 w_{xx} \equiv w_{tt}$$

Next,

$$w(x, 0) = R_t(x, 0) = \varphi(x)$$

$$w_t(x, 0) = R_{tt}(x, 0) = G_{tt}(x, 0) * \varphi(x) = 0 * \varphi(x) = 0$$

If the function $\varphi(z)$ is continuous, then $w(x, t)$ can be written in the form

$$w(x, t) = R_t(x, t) = \frac{\partial}{\partial t}\left\{\frac{1}{2a}\int_{x-at}^{x+at} \varphi(z)\,dz\right\}$$

$$= \frac{\varphi(x-at)+\varphi(x+at)}{2}$$

i.e.

$$w(x, t) = \frac{\varphi(x-at)+\varphi(x+at)}{2}$$

The solution of the arbitrary Cauchy problem

$$a^2u_{xx} = u_{tt}, \qquad u(x, 0) = \varphi(x), \qquad u_t(x, 0) = \psi(x)$$

where $\varphi(x)$ is a continuous function and $\psi(x)$ is integrable (this includes piecewise-continuous functions) will be the sum

$$u = v+w = G(x, t) * \psi(x)+G_t(x, t) * \varphi(x)$$

or

$$u(x, t) = \frac{\varphi(x-at)+\varphi(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \psi(z)\,dz$$

It follows that, in this case, the solution can again be deduced from d'Alembert's formula. Its derivatives are then treated as the derivatives of generalised functions and coincide with the ordinary derivatives when the latter exist.

We note that the expression given by (8) therefore yields the solution of the Cauchy problem even for arbitrary generalised initial functions $\varphi(x)$ and $\psi(x)$.

### 3.3 VIBRATIONS OF A LOADED INFINITE STRING

**3.3.1** Now that we know how to obtain the solution of Cauchy's problem for the homogeneous wave equation (2), we can readily establish the solution of this problem for the inhomogeneous wave

equation (1). The method is the same for all linear hyperbolic equations and we shall therefore confine our attention to the general equation

$$\operatorname{div}(k\,\nabla u) - qu + f(M, t) = \rho u_{tt} \tag{11}$$

where $k$, $q$ and $\rho$ are known functions of the coordinates of the point $M$.

Thus, suppose that it is required to solve Cauchy's problem for (11) subject to the initial conditions

$$u(M, 0) = \varphi(M), \qquad u_t(M, 0) = \psi(M) \tag{12}$$

We shall split this problem into two parts:

1. Cauchy's problem for the homogeneous equation

$$\operatorname{div}(k\,\nabla v) - qv = \rho v_{tt} \tag{13}$$

subject to the initial conditions

$$v(M, 0) = \varphi(M), \qquad v_t(M, 0) = \psi(M) \tag{14}$$

and

2. Cauchy's problem for the original equation

$$\operatorname{div}(k\,\nabla w) - qw + f(M, t) = \rho w_{tt} \tag{11'}$$

subject to the initial conditions

$$w(M, 0) = 0, \qquad w_t(M, 0) = 0 \tag{15}$$

It is evident that $u = v + w$.

Let us suppose that we know how to solve Cauchy's problem (13)–(14). The solution of Cauchy's problem (11′)–(15) can then be determined as follows. Consider the function $W(M, t, \tau)$ which satisfies the homogeneous Equation (13) and the initial conditions

$$W_{t-\tau} = 0, \qquad W_t|_{t=\tau} = \frac{f(M, \tau)}{\rho(M)} \tag{16}$$

By hypothesis we know how to solve this problem. The required solution of Cauchy's problem (2) will then be of the form

$$w(M, t) = \int_0^t W(M, t, \tau)\,d\tau \tag{17}$$

In fact,

$$w_t(M, t) = W|_{t=\tau} + \int_0^t W_t(M, t, \tau)\,d\tau$$

and using the first of the conditions in (16) we obtain

$$w_t(M, t) = \int_0^t W_t(M, t, \tau)\, d\tau \tag{18}$$

From (17) and (18) it follows at once that $w(M, t)$ satisfies the initial conditions. Differentiating (18) with respect to $t$ once again and using the second of the conditions in (16), we obtain

$$w_{tt} = W_t|_{\tau=t} + \int_0^t W_{tt}(M, t, \tau)\, d\tau = \frac{f(M, t)}{\rho(M)} + \int_0^t W_{tt}(M, t, \tau)\, d\tau$$

Consequently,

$$\rho w_{tt} = f(M, t) + \int_0^t \rho W_{tt}\, d\tau \tag{19}$$

Let us evaluate the quantity $\operatorname{div}(k\nabla w) - qw$. The operation $\operatorname{div}(k\nabla)$ can, of course, be carried out under the integral sign. We have

$$\operatorname{div}(k\nabla w) - qw = \int_0^t \{\operatorname{div}(k\nabla W) - qW\}\, d\tau \tag{20}$$

Since the function $W(M, t, \tau)$ is a solution of (13), it follows from (19) and (20) that the function $w(M, t)$ defined by (17) is a solution of (11′) and, consequently, a solution of Cauchy's problem (2).

**3.3.2** Let us now use the above method to solve Equation (1). It is required to solve the Cauchy problem

$$a^2 u_{xx} + f(x, t) = u_{tt}$$

$$u(x, 0) = \varphi(x), \qquad u_t(x, 0) = \psi(x)$$

We shall split this problem into two parts.

1. Cauchy's problem for the homogeneous equation with given initial conditions

$$a^2 v_{xx} = v_{tt},$$

$$v(x, 0) = \varphi(x), \qquad v_t(x, 0) = \psi(x)$$

2. Cauchy's problem for the given equation with zero initial values

$$a^2 w_{xx} + f(x, t) = w_{tt},$$
$$w(x, 0) = 0, \qquad w_t(x, 0) = 0$$

We then have $u = v + w$.

The function $v(x, t)$ can be obtained from the d'Alembert formula

$$v(x, t) = \frac{\varphi(x-at)+\varphi(x+at)}{2} + \frac{1}{2a} \int_{x-at}^{x+at} \psi(z)\, dz$$

In accordance with the foregoing discussion

$$w(x, t) = \int_0^t W(x, t, \tau)\, d\tau$$

where $W(x, t, \tau)$ is a solution of Cauchy's problem

$$a^2 W_{xx} = W_{tt},$$
$$W|_{t=\tau} = 0, \qquad W_t|_{t=\tau} = f(x, \tau)$$

and, consequently, can be written in accordance with the d'Alembert formula

$$W(x, t, \tau) = \frac{1}{2a} \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(z, \tau)\, dz$$

Therefore,

$$w(x, t) = \frac{1}{2a} \int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(z, \tau)\, dz\, d\tau$$

## 3.4 SOLUTION OF BOUNDARY-VALUE PROBLEMS ON A SEMI-INFINITE STRAIGHT LINE

**3.4.1** We shall now consider boundary-value problems on a semi-infinite line. To begin with, we shall prove the following two lemmas.

*Lemma 1* If in Cauchy's problem (2)–(3) the initial functions $\varphi(x)$ and $\psi(x)$ are odd with respect to $x = 0$, the solution of this problem at $x = 0$ is

$$u(0, t) \equiv 0$$

*Proof* Substituting $x = 0$ in d'Alembert's formula which gives a solution of Cauchy's problem (2)–(3), we obtain

$$u(0, t) = \frac{\varphi(-at)+\varphi(at)}{2} + \frac{1}{2a} \int_{-at}^{at} \psi(z)\, dz$$

Since $\varphi(x)$ is an odd function, it follows that $\varphi(-at) \equiv -\varphi(at)$. Therefore, $\varphi(-at)+\varphi(at) \equiv 0$. Since $\psi(x)$ is odd, the integral

$$\int_{-at}^{at} \psi(z)\, dz$$

is also zero. Consequently, $u(0, t) \equiv 0$.

*Lemma 2* If in Cauchy's problem (2)–(3) the initial functions $\varphi(x)$ and $\psi(x)$ are even with respect to $x = 0$, the derivative $u_x(x, t)$ of the solution of this problem at $x = 0$

$$u_x(0, t) \equiv 0$$

The proof of this lemma is similar to that given above. It is based on the fact that the derivative $\varphi'(x)$ is odd.

**3.4.2** Consider the homogeneous boundary-value problem

$$\begin{aligned} &a^2 u_{xx} = u_{tt} \\ &u(x, 0) = \varphi(x), \qquad u_t(x, 0) = \psi(x), \qquad 0 \leqslant x < \infty \\ &u(0, t) = 0 \end{aligned} \tag{21}$$

We shall assume that $\varphi(0) = \psi(0) = 0$. To solve this problem we cannot use d'Alembert's formula directly because the difference $x-at$ in this expression may be negative, and the initial functions $\varphi(x)$ and $\psi(x)$ are not defined for negative values of the argument, in accordance with (21).

We shall proceed as follows. We shall continue the functions $\varphi(x)$ and $\psi(x)$ along the negative part of the $x$ axis by defining new functions

$$\varphi_1(x) = \begin{cases} \varphi(x), & x \geqslant 0, \\ -\varphi(-x), & x < 0, \end{cases} \qquad \psi_1(x) = \begin{cases} \psi(x), & x \geqslant 0 \\ -\psi(-x), & x < 0 \end{cases}$$

The function

$$u(x, t) = \frac{\varphi_1(x-at)+\varphi_1(x+at)}{2} + \frac{1}{2a} \int_{x-at}^{x+at} \psi_1(z)\, dz$$

will then be a solution of the boundary-value problem. In fact, it satisfies the homogeneous wave equation because it is a superposition of forward and reverse waves. It satisfies the boundary condition in view of Lemma 1. Let us verify the initial conditions:

$$\begin{aligned} u(x, 0) &= \frac{\varphi_1(x)+\varphi_1(x)}{2} + \frac{1}{2a}\int_x^x \psi_1(z)\,dz \\ &= \varphi_1(x) = \varphi(x), \qquad x \geqslant 0 \\ u_t(x, 0) &= \frac{-a\varphi_1'(x)+a\varphi_1'(x)}{2} + \frac{1}{2a}[a\psi_1(x)+a\psi_1(x)] \\ &= \psi_1(x) = \psi(x), \qquad x \geqslant 0 \end{aligned}$$

It follows that the initial conditions are also satisfied.

The boundary-value problem

$$\begin{aligned} &a^2u_{xx} = u_{tt}, \\ &u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \qquad 0 < x < \infty \qquad (22) \\ &u_x(0, t) = 0 \end{aligned}$$

can be solved in a similar way except that the functions $\varphi(x)$ and $\psi(x)$ are now continued as even functions along the negative part of the straight line.

## 3.5 REFLECTION OF WAVES AT FIXED AND FREE ENDS

The solution of the boundary-value problems (21) and (22) can be written in the form (6):

$$u(x, t) = \Phi(x-at)+F(x+at)$$

We shall interpret $u$ as a displacement. The displacement due to the forward wave is constant along the characteristic $x-at = c_1$, i.e. $\Phi(x-at) = \Phi(c_1)$. Along the characteristic $x+at = c_2$ the displacement due to the reverse wave is also constant, i.e. $F(x+at) = F(c_2)$. It follows that the displacements propagate along the characteristic curves.

Let us draw the two characteristics $x-at = x-at_0$ and $x+at = x_0+at_0$ through the point $(x_0, t_0)$ in the $(x, t)$ plane. They will intersect the $x$ axis at points $-x_1$ and $x_2$, respectively (Fig. 3.3).

The displacement $u(x_0, t_0)$ at point $x_0$ and time $t_0$ consists of the displacement due to the reverse wave which arrives from the point $x_2$ and the displacement due to the forward wave which arrives from the point $-x_1$. However, at the point $-x_1$ there can be no displacement at $t = 0$ because the initial conditions in the

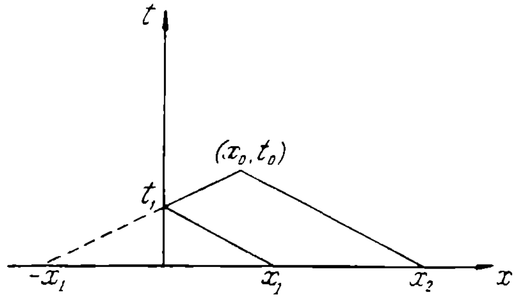



Fig. 3.3

problems defined by (21) and (22) are prescribed only on the semi-infinite straight line for which $x > 0$. However, from the boundary condition for the problem defined by (21) it follows that

$$\Phi(-z) \equiv -F(z)$$

Consequently, $\Phi(-x_1) = -F(x_1)$. It follows that instead of the forward wave arriving from the point $-x_1$ we can consider the reverse wave which has left the symmetric point $x_1$ at time $t = 0$. This reverse wave will reach the point $x = 0$ in a time $t_1$. From $t = t_1$ onwards, it must be replaced by the forward wave which has left the point $x = 0$ at time $t = t_1$ and has propagated with displacement $-\Phi(-x_1)$. It follows that when the boundary condition $u(0, t) = 0$ is satisfied at $x = 0$ we have the phenomenon of reflection in which the magnitude of the displacement is conserved but its sign changes.

It can be shown in a similar way that when the boundary condition $u_x(0, t) = 0$ is satisfied at $x = 0$, we have the phenomenon of reflection with the conservation of the magnitude and of the sign of the displacement.

The method of characteristics can also be used to construct solutions of homogeneous boundary-value problems on a finite segment subject to boundary conditions of types I and II. To be specific, consider the first boundary value problem

$$\begin{aligned} &a^2 u_{xx} = u_{tt} \\ &u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \qquad 0 \leqslant x \leqslant l \\ &u(0, t) = 0, \qquad u(l, t) = 0, \qquad t \geqslant 0 \end{aligned} \tag{23}$$

To obtain the solution let us continue the initial functions $\varphi(x)$ and $\psi(x)$ along the entire line and assume that they are odd with respect to $x = 0$ and $x = l$. Let $\varphi_2(x)$ and $\psi_2(x)$ represent the functions continued in this way. The function

$$u(x, t) = \frac{\varphi_2(x-at)+\varphi_2(x+at)}{2} + \frac{1}{2a} \int_{x-at}^{x+at} \psi_2(z)\,dz$$

will then be a solution of the boundary-value problem. This function satisfies the boundary conditions in view of Lemma 1. The initial conditions can be verified directly as in the case of the semi-infinite line.

Note that if the function $\varphi(x)$ is odd (even) with respect to two points, say, $x = 0$ and $x = l$, then it is periodic with period $2l$. In fact, since $\varphi(x)$ is odd with respect to $x = l$, we have the identity $\varphi(l-z) \equiv -\varphi(l+z)$. Substituting $z = x+l$, we obtain $\varphi(-x) \equiv -\varphi(x+2l)$. Since $\varphi(-x) \equiv -\varphi(x)$, it follows that $\varphi(x+2l) \equiv \varphi(x)$. The initial functions $\varphi(x)$ and $\psi(x)$ can therefore be continued to the segment $(-l, 0)$ as odd functions and then periodically with a period $2l$ along the entire straight line.

## 3.6 PROPAGATION OF THE BOUNDARY VALUE ALONG A SEMI-INFINITE STRAIGHT LINE

Consider the inhomogeneous boundary-value problem on a semi-infinite straight line

$$\begin{aligned} &a^2 u_{xx} = u_{tt}, \\ &u(x, 0) = \varphi(x), \qquad u_t(x, 0) = \psi(x), \qquad x > 0 \\ &u(0, t) = \mu(t), \qquad t > 0 \end{aligned}$$

It can be split into two problems, namely:

1. the homogeneous boundary-value problem

$$\begin{aligned} &a^2 v_{xx} = v_{tt}, \\ &v(x, 0) = \varphi(x), \qquad v_t(x, 0) = \psi(x), \qquad x > 0 \\ &v(0, t) = 0 \end{aligned}$$

2. the problem involving propagation of the boundary value

$$\begin{aligned} &a^2 w_{xx} = w_{tt}, \\ &w(x, 0) = 0, \qquad w_t(x, 0) = 0 \\ &w(0, t) = \mu(t), \qquad t > 0 \end{aligned}$$

We then have $u = v+w$.

We already know how to solve the problem for $v(x, t)$. The propagation of the boundary value will now be considered. Since the only reason for the appearance of a perturbation is the situation at the boundaries, we shall seek the solution in the form of the forward wave

$$w(x, t) = \Phi(x-at)$$

From the initial conditions we find that

$$w(x, 0) = \Phi(x) \equiv 0, \qquad x > 0$$

It is evident that $w_t(x, 0) = -a\Phi'(x) \equiv 0$ will also be satisfied for $x > 0$. From the boundary condition we find

$$\Phi(-at) = \mu(t), \qquad t > 0$$

and therefore

$$\Phi(z) = \begin{cases} 0, & z > 0 \\ \mu\left(-\dfrac{z}{a}\right), & z < 0 \end{cases}$$

or $\Phi(z) = \eta(-z/a)\mu(-z/a)$, where $\eta(\xi)$ is the unit step function (equal to unity for $\xi > 0$ and zero for $\xi < 0$). Consequently,

$$w(x, t) = \eta\left(t-\frac{x}{a}\right)\mu\left(t-\frac{x}{a}\right)$$

The propagation of boundary conditions of type II can be discussed in a similar way: $u_x(0, t) = \nu(t)$.

Using the phenomenon of reflection considered above, we can readily solve the problem of the propagation of the boundary conditions of type I or type II along a finite segment.

The method of characteristics can also be used to solve a number of other problems, which can be formulated in terms of one-dimensional inhomogeneous wave equation. However, the problems which we have already discussed illustrate all the main features of the method and we shall therefore confine our attention to them.

## 3.7 VIBRATIONS OF AN INFINITE MEDIUM. POISSON'S FORMULA

Many problems leading to the two- or three-dimensional wave equations can be reduced to problems of the form already discussed. Let us consider some of them.

**3.7.1** Cauchy's problem for the homogeneous wave equation in three-dimensional space:

$$a^2 \nabla^2 u = u_{tt} \tag{24}$$

$$u(M, 0) = \varphi(M), \qquad u_t(M, 0) = \psi(M) \tag{25}$$

To solve this problem consider the auxiliary function $\bar{u}(r, t)$ which is the average of the required solution over a sphere $S_M^r$ centred on the point $M$ and having a radius $r$:

$$\bar{u}(r, t) = \frac{1}{4\pi r^2} \iint\limits_{S_M^r} u(P, t)\,\mathrm{d}\sigma_p \tag{26}$$

where $P$ denotes the variable integration point.

Let $\mathrm{d}\omega$ be an element of solid angle subtended by the area $\mathrm{d}\sigma$ at $M$ so that $\mathrm{d}\sigma = r^2\,\mathrm{d}\omega$. It follows that we can also write

$$\bar{u}(r, t) = \frac{1}{4\pi} \iint\limits_{S_M^r} u(P, t)\,\mathrm{d}\omega \tag{27}$$

Using the mean value theorem in Equation (26) and letting $r$ tend to zero, we obtain

$$\bar{u}(0, t) = u(M, t) \tag{28}$$

It follows that to find the function $u(M, t)$, it is sufficient to find the function $\bar{u}(r, t)$. To do this we shall require the following lemma.

*Lemma*

$$\overline{\nabla^2 u} = \nabla_r^2(\bar{u})$$

where on the left-hand side the Laplacian $\nabla^2 u$ is evaluated with respect to the coordinates of $M$, and the right-hand side is evaluated with respect to the coordinates of point $r$. Henceforth, we shall omit the subscript $r$.

*Proof* Let $D_M^r$ be a region bounded by the spherical surface $S_M^r$. Using Gauss's theorem we have

$$\iiint\limits_{D_M^r} \nabla^2 u\,\mathrm{d}\tau = \iint\limits_{S_M^r} \frac{\partial u}{\partial n}\,\mathrm{d}\sigma = \iint\limits_{S_M^r} \frac{\partial u}{\partial r}\,\mathrm{d}\sigma = r^2 \iint\limits_{S_M^r} \frac{\partial u}{\partial r}\,\mathrm{d}\omega$$

$$= r^2 \frac{\partial}{\partial r} \iint\limits_{S_M^r} u\,\mathrm{d}\omega$$

Applying (27) to the last integral, we obtain

$$\iiint\limits_{D_M^r} \nabla^2 u \, \mathrm{d}\tau = 4\pi r^2 \frac{\partial \bar{u}}{\partial r}$$

On the other hand,

$$\iiint\limits_{D_M^r} \nabla^2 u \, \mathrm{d}\tau = \int_0^r \Big(\iint\limits_{S_M^\rho} \nabla^2 u \, \mathrm{d}\sigma\Big) \mathrm{d}\rho = \int_0^r (4\pi\rho^2 \overline{\nabla^2 u}) \, \mathrm{d}\rho$$

and consequently,

$$\int_0^r \rho^2 \overline{\nabla^2 u} \, \mathrm{d}\rho = r^2 \frac{\partial \bar{u}}{\partial r}$$

Differentiating this with respect to $r$ we obtain

$$\overline{\nabla^2 u} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \bar{u}_r) = \nabla^2(\bar{u})$$

This completes the proof of the lemma.

Let us now suppose that the solution of the problem (24)–(25) exists. Averaging the identity

$$a^2 \nabla^2 u \equiv u_{tt}$$

over the sphere $S_M^r$ and using the above lemma, we obtain

$$a^2 \nabla^2 \bar{u} \equiv \bar{u}_{tt}$$

or

$$a^2 \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \bar{u}_r) = \bar{u}_{tt}$$

or

$$a^2 (r \bar{u}_{rr} + 2\bar{u}_r) \equiv r \bar{u}_{tt}$$

Substituting $v = r\bar{u}$, the last result can be rewritten in the form

$$a^2 v_{rr} \equiv v_{tt}$$

It follows that the function $v(r, t)$ satisfies the one-dimensional wave equation.

If we take the average of (25), we obtain

$$\begin{aligned} \bar{u}(r, 0) &= \bar{\varphi}(r) = \frac{1}{4\pi r^2} \iint\limits_{S_M^r} \varphi(P)\, d\sigma \\ \bar{u}_t(r, 0) &= \bar{\psi}(r) = \frac{1}{4\pi r^2} \iint\limits_{S_M^r} \psi(P)\, d\sigma \end{aligned} \tag{29}$$

Let

$$\varphi_1(r) = r\bar{\varphi}(r) \quad \text{and} \quad \psi_1(r) = r\bar{\psi}(r)$$

It is evident that

$$v(r, 0) = \varphi_1(r), \qquad v_t(r, 0) = \psi_1(r), \qquad v(0, t) = 0$$

Therefore, for $v(r, t)$ we have the following problem on a semi-infinite straight line

$$\begin{aligned} &a^2 v_{rr} = v_{tt} \\ &v(r, 0) = \varphi_1(r), \qquad v_t(r, 0) = \psi_1(r) \\ &v(0, t) = 0 \end{aligned}$$

To solve this problem, the initial functions $\varphi_1(r)$ and $\psi_1(r)$ should, in accordance with Section 3.4, be continued as odd functions along the semi-infinite line $(-\infty, 0)$, and the d'Alembert formula should be used for the continued functions $\varphi_2(r)$ and $\psi_2(r)$. The continued functions $\bar{\varphi}(r)$ and $\bar{\psi}(r)$ will be odd [we shall retain the previous notation for the continued function, namely, $\bar{\varphi}(r)$ and $\bar{\psi}(r)$]. The solution of the problem for $v(r, t)$ will be of the form

$$v(r, t) = \frac{\varphi_2(r+at)+\varphi_2(r-at)}{2} + \frac{1}{2a} \int\limits_{r-at}^{r+at} \psi_2(z)\, dz$$

Consequently,

$$\bar{u}(r, t) = \frac{v}{r} = \frac{\varphi_2(r+at)+\varphi_2(r-at)}{2r} + \frac{1}{2ar} \int\limits_{r-at}^{r+at} \psi_2(z)\, dz$$

If we substitute $r = 0$ in this formula, we obtain $\bar{u}(0, t) = 0/0$.

The function $\bar{u}(0, t)$ can be evaluated with the aid of l'Hôpital's rule (bearing in mind the definition of $\varphi_2$ and $\psi_2$):

$$\bar{u}(0, t) = \frac{1}{2}\{\bar{\varphi}(at)+\bar{\varphi}(-at)+at\bar{\varphi}'(at)-at\bar{\varphi}'(-at)\}$$

$$+\frac{1}{2a}\{at\bar{\psi}(at)+at\bar{\psi}(-at)\}$$

Since the functions $\bar{\varphi}(z)$ and $\bar{\psi}(z)$ are even, whereas $\bar{\varphi}'(z)$ is odd, it follows that

$$\bar{u}(0, t) = \bar{\varphi}(at)+at\bar{\varphi}'(at)+t\bar{\psi}(at) = \frac{\mathrm{d}}{\mathrm{d}t}\{t\bar{\varphi}(at)\}+t\bar{\psi}(at) \quad (30)$$

Using (28), (29) and (30), we obtain Poisson's formula for the required solution of Cauchy's problem (24)–(25):

$$u(M, t) = \frac{1}{4\pi a}\frac{\partial}{\partial t}\iint\limits_{S_M^{at}}\frac{\varphi(P)\,\mathrm{d}\sigma}{at}+\frac{1}{4\pi a}\iint\limits_{S_M^{at}}\frac{\psi(P)}{at}\,\mathrm{d}\sigma \quad (31)$$

Thus, if we assume the existence of the solution of Cauchy's problem for the three-dimensional space, we find that it must be given by (31). Consequently, this is the only solution.

**3.7.2** We can now solve Cauchy's problem in three-dimensional space for the inhomogeneous wave equation

$$a^2\nabla^2 u+f(M, t) = u_{tt}$$
$$u(M, 0) = \varphi(M), \qquad u_t(M, 0) = \psi(M)$$

We can split it into two subsidiary problems:

1. $$a^2\nabla^2 v = v_{tt}$$
$$v(M, 0) = \varphi(M), \qquad v_t(M, 0) = \psi(M)$$

The solution of this problem is given by Poisson's formula.

2. $$a^2\nabla^2 w+f(M, t) = w_{tt}$$
$$w(M, 0) = 0, \qquad w_t(M, 0) = 0$$

It is evident that $u = v+w$.

Problem 2 can be solved by the method described in Section 3.3. In particular, we must first solve the auxiliary problem

$$a^2\nabla^2 W = W_{tt}$$
$$W|_{t=\tau} = 0, \qquad W_t|_{t=\tau} = f(M, \tau)$$

and then from Poisson's formula we have

$$W(M, t, \tau) = \frac{1}{4\pi a} \iint\limits_{S_M^{a(t-\tau)}} \frac{f(P, \tau)}{a(t-\tau)} \, d\sigma$$

As was shown above,

$$w(M, t) = \int_0^t W(M, t, \tau) \, d\tau$$

or

$$w(M, t) = \frac{1}{4\pi a} \int_0^t \left[ \iint\limits_{S_M^{a(t-\tau)}} \frac{f(P, \tau)}{a(t-\tau)} \, d\sigma \right] d\tau$$

Substituting $a(t-\tau) = r$ in the outer integral, we obtain

$$w(M, t) = \frac{1}{4\pi a^2} \int_0^{at} \left[ \iint\limits_{S_M^r} \frac{f\left(P, t - \dfrac{r}{a}\right)}{r} \, d\sigma \right] dr$$

where $r$ is the distance between the integration point $P$ and the point $M$, i.e. $r = r_{MP}$. This integral can, clearly, be written as an integral over the region $D_M^{at}$ bounded by the sphere $S_M^{at}$:

$$w(M, t) = \frac{1}{4\pi a^2} \iiint\limits_{D_M^{at}} \frac{f\left(P, t - \dfrac{r}{a}\right)}{r} \, dv \tag{32}$$

If the external driving force $f(M, t)$ is different from zero only at the single point $M_0$, where it is equal to $f(t)$, the wave equation can be written in the form

$$a^2 \nabla^2 u + f(t) \delta(M, M_0) = u_{tt}$$

where $\delta(M, M_0)$ is the $\delta$ function with a singularity at $M_0$ (see Appendix).

The solution of this equation which has zero initial values [and is therefore due only to the point force $f(t)$] can be written in accordance with Equation (32) in the form

$$u(M, t) = \frac{1}{4\pi a^2} \iiint\limits_{D_M^{at}} \frac{f\left(t - \dfrac{r}{a}\right) \delta(P, M_0)}{r} \, dv$$

Using the fundamental property of the $\delta$ function, and remembering that for any continuous function $\varphi(M)$

$$\iiint\limits_D \varphi(P)\,\delta(P, M_0)\,\mathrm{d}v = \begin{cases} 0 & \text{if } M_0 \notin D \\ \varphi(M_0) & \text{if } M_0 \in D \end{cases}$$

we obtain

$$u(M, t) = \begin{cases} 0, & \text{where} \quad at < r_{MM_0} \\ \dfrac{1}{4\pi a^2}\dfrac{1}{r_{MM_0}} f\left(t - \dfrac{r_{MM_0}}{a}\right), & \text{where} \quad at > r_{MM_0} \end{cases} \tag{33}$$

**3.7.3** The solution of Cauchy's problem for the homogeneous wave equation in two-dimensional space

$$a^2 \nabla^2 u = u_{tt}$$

$$u(x, y, 0) = \varphi(x, y), \qquad u_t(x, y, 0) = \psi(x, y) \tag{34}$$

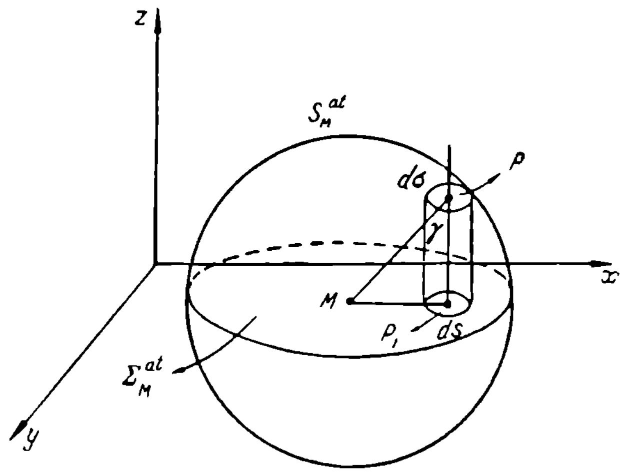



*Fig. 3.4*

can also be obtained from Poisson's formula. In point of fact, if the functions $\varphi(P)$ and $\psi(P)$ in Equation (31) are independent of $z$, then the integrals over the surface $S_M^{at}$ of the sphere can be reduced to integrals over the major circle $\sum_M^{at}$ of this sphere in the $(x, y)$ plane (Fig. 3.4). The integral over the upper half of $S_M^{at}$ is equal to

$$\iint\limits_{\text{upper } S_M^{at}} \frac{\varphi(P)}{at}\,\mathrm{d}\sigma = \iint\limits_{\Sigma_M^{at}} \frac{\varphi(P_1)}{at}\,\frac{\mathrm{d}\sigma}{\cos\gamma}$$

where $\gamma$ is the angle between the normals to the $(x, y)$ plane and to the sphere $S_M^{at}$ at the point $P$. It is evident that

$$\cos\gamma = \frac{|PP_1|}{|MP|} = \frac{\sqrt{(at)^2-|P_1M|^2}}{at}$$

$$= \frac{\sqrt{(at)^2-(x-\xi)^2-(y-\eta)^2}}{at}$$

where $\xi$, $\eta$ are the coordinates of the point $P_1$ which is the projection of $P$ or, to the $(x, y)$ plane, and $(x, y)$ are the coordinates of the point of observation $M$. Therefore,

$$\iint\limits_{\text{upper } S_M^{at}} \frac{\varphi(p)}{at}\,\mathrm{d}\sigma = \iint\limits_{\Sigma_M^{at}} \frac{\varphi(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta}{\sqrt{(at)^2-(x-\xi)^2-(y-\eta)^2}}$$

and, similarly,

$$\iint\limits_{\text{lower } S_M^{at}} \frac{\varphi(P)}{at}\,\mathrm{d}\tau = \iint\limits_{\Sigma_M^{at}} \frac{\varphi(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta}{\sqrt{a^2t^2-(x-\xi)^2-(y-\eta)^2}}$$

Using a similar transformation for the second integral in Poisson's formula, we obtain a solution of Cauchy's problem (34) in the form

$$u(x, y, t) = \frac{1}{2\pi a}\frac{\partial}{\partial t}\iint\limits_{\Sigma_M^{at}} \frac{\varphi(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta}{\sqrt{a^2t^2-(x-\xi)^2-(y-\eta)^2}}$$

$$+\frac{1}{2\pi a}\iint\limits_{\Sigma_M^{at}} \frac{\psi(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta}{\sqrt{a^2t^2-(x-\xi)^2-(y-\eta)^2}} \tag{35}$$

We are now in a position to solve Cauchy's problem in two-dimensional space and for the inhomogeneous equation. It can be reduced to the problem just considered and to Cauchy's problem for the inhomogeneous equation with zero initial values. The latter problem can be solved by the method described in Section 3.3. We shall not repeat all the steps leading to this solution and shall merely quote the final result

$$w(x, y, t) = \frac{1}{2\pi a}\int_0^t\left(\iint\limits_{\Sigma_M^{a(t-\tau)}} \frac{f(\xi,\eta,\tau)\,\mathrm{d}\xi\,\mathrm{d}\eta}{\sqrt{a^2(t-\tau)^2-(x-\xi)^2-(y-\eta)^2}}\right)\mathrm{d}\tau \tag{36}$$

We leave it as an exercise for the reader to obtain the solution due to a point force $f(t)$.

## 3.8 PHYSICAL INTERPRETATION OF POISSON'S FORMULA

Let us return now to Poisson's formula. Suppose the initial functions $\varphi(M)$ and $\psi(M)$ are non-zero only within a domain $D$ bounded by the surface $S$ (Fig. 3.5). Let us observe the state of the medium

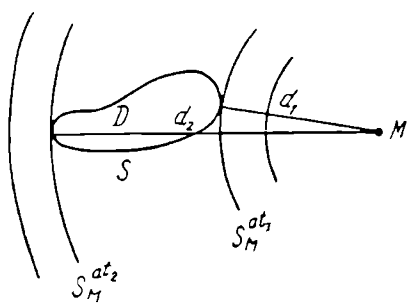



*Fig. 3.5*

at a fixed point $M$. For sufficiently small $t$ the surface $S_M^{at}$ of the sphere centred on $M$ will not cut the domain $D$. The integrals on the right-hand side of Poisson's formula will therefore be zero. Consequently, for these values of $t$ we have $u(M, t) = 0$ (the disturbance has not reached the point $M$). Let $d_1$ be the distance of the nearest point on the surface $S$ from $M$, and let $d_2$ be the distance of the most distant point on $S$ from $M$. For $t \in (t_1, t_2)$ $(t_1 = d_1/a,$ $t_2 = d_2/a)$ the surface $S_M^{at}$ of the sphere will cut the domain $D$. Therefore, the integrals in Poisson's formula will not be zero and, therefore, for these values of $t$ we have $u(M, t) \neq 0$. For $t > t_2$ the sphere $S_M^{at}$ will not cut the domain $D$ and, consequently, $u(M, t)$ will again be zero.

Let us suppose now that from each point in the domain $D$ disturbances are emitted and propagate with velocity $a$ in all directions (i.e. the Huygens principle). The above changes in the function $u(M, t)$ with time can then be physically interpreted as follows. For $t < t_1$ the disturbances have not reached the point $M$. At time $t = t_1$ the leading edge of the wave reaches the point $M$ and the wave passes $M$ for $t_1 \leqslant t \leqslant t_2$. At time $t = t_2$ the trailing edge of the wave passes $M$ and from this moment onwards the point $M$ remains at rest.

Suppose that the initial functions $\varphi(x, y)$ and $\psi(x, y)$ for the two-dimensional case are non-zero only in the domain $D$ bounded

by the curve $S$ (Fig. 3.6). For $t < t_1$ the circle $\sum_M^{at}$ does not contain points belonging to $D$, and therefore the integrals in (35) are zero so that $u(x, y, t) = 0$. For any $t > t_1$ the circle $\sum_M^{at}$ contains the domain $D$ or part of it and, therefore, $u(x, y, t) \neq 0$ for these values of $t$.

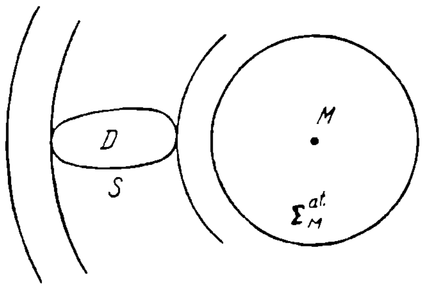



*Fig. 3.6*

The times $t_1$ and $t_2$ are defined as before. Thus, in the two-dimensional case, the wave has a leading edge (it reaches the point $M$ at time $t = t_1$) but no trailing edge. Huygens' principle is then invalid. This is readily understood if we recall that the two-dimensional case which we have been considering is, in fact, a three-dimensional problem in which the region of non-zero values of the initial functions $\varphi(M)$ and $\psi(M)$ is an infinite cylinder, whose generators are parallel to the $z$ axis. It is evident that the spherical surface $S_M^{at}$ will cut the cylinder for any $t > t_1$, and therefore the integrals in Poisson's formula will be non-zero for all $t > t_1$.

## PROBLEMS

1. An infinite string is excited by a triangular initial deflection in the range $(c, 2c)$ with corners at the points $c$, $3c/2$ and $2c$. Draw the profile of the string for times $t_k = ck/2a$, where $k = 1, 2, 3$.

2. Solve Problem 1 if the initial deflection is in the form of a triangle lying in the ranges $(-2c, -c)$ and $(c, 2c)$ with corners at the points $-2c$, $-1.5c$, $-c$, $c$, $1.5c$, $2c$.

3. An infinite string is given an initial transverse velocity $v_0 = \text{const}$ in the range $-c \leqslant x \leqslant c$. Determine the vibrations of the string. Draw the profile of the string for times $t_k = ck/2a$ $(k = 1, 2, 3)$.

4. A semi-infinite string with one end rigidly fixed is excited by an initial deflection which is no-zero only on the segment $(c, 3c)$, where it takes the form of a broken line with corners at the point $c$, $2c$, $3c$. Draw the profile of the string for times $t_k = ck/2a$ $(k= 2, 4, 6)$.

5. At the initial time $t = 0$ a semi-infinite string with one end rigidly fixed is given a transverse impulse at the point $x = x_0$. Determine the vibrations of the string.

6. An infinite elastic rod consists of two semi-infinite homogeneous rods, joined at the point $x = 0$. The densities and elastic constant of the two materials are $\rho_1, E_1$ and $\rho_2, E_2$, respectively. Suppose that the wave $u_1(x, t) = f(t - x/a_1)$ travels along the rod from the region $x < 0$. Find the reflected and refracted waves. When will they exist? Investigate the solution for $E_2 \to 0$ and $E_2 \to \infty$.

7. A constant e.m.f. $E_0$ is applied for a sufficiently long interval of time to the end $x = 0$ of a semi-infinite non-distorting transmission line ($GL = CR$), so that a stationary distribution of voltage and current is set up in the conductor. At time $t = 0$ the end of the conductor is earthed through a lumped resistance $R_0$. Find the voltage and current in the conductor for $t > 0$.

8. The ends $x = 0$, $x = l$ of a string are rigidly fixed. The initial deflection is $u(x, 0) = A \sin (\pi/l)x$ $(0 \leqslant x \leqslant l)$ and the initial velocity is zero. Draw the profile of the string for times $t_k = lk/2a$ $(k = 1, 2, 4)$.

9. Determine the vibrations of an infinite string under the action of a localised transverse force $F(t)$ $(t > 0)$ if the point of application of the force moves along the string at a constant velocity $v_0$ from the position $x = 0$ and $v_0 < a$.

10. An e.m.f. $E = f(t)$ is applied at time $t = 0$ to the end $x = 0$ of a semi-infinite conductor with negligible resistance and leakage. Find the potential distribution $u(x, t)$ in the conductor.

11. A capacitance $C_0$ charged to a potential difference $V$ is discharged at time $t = 0$ into an infinite conductor having an inductance $L$ and a capacitance $C$. Determine the current in the conductor.

12. A compression $S_0 = (\rho - \rho_0)/\rho_0$ is produced at time $t = 0$ in a gas at rest. The disturbance is localised within a volume bounded by a given surface $\sigma$. Find the compression $S(M, t)$ as function of the area $\sigma_t$ of the part of the surface of the sphere $S_M^{at}$ lying inside $\sigma$.

13. Which linear equations with constant coefficients of the form $a_{11}u_{xx} + 2a_{12}u_{xt} + a_{22}u_{tt} + b_1 u_x + b_2 u_t + c\,u = 0$ have solutions in the form of arbitrary travelling waves $f(x - at)$, where $a = \text{const}$? (There is no dispersion.)

14. Which equations of the form given in Problem 13 have solutions in the form of arbitrary damped travelling waves of the form $e^{-\mu t} f(x - at)$?

# 4

# Separation of Variables (Fourier Method)

Typical problems which can be solved by the method of separation of variables are boundary-value problems for hyperbolic and parabolic equations in bounded regions. The principle of the method is best illustrated by the simplest case, i.e. the case of homogeneous boundary-value problems. We shall consider in parallel boundary-value problems for hyperbolic and parabolic equations.

## 4.1 PRINCIPLE OF THE METHOD. EIGENFUNCTIONS AND EIGENVALUES

**4.1.1** Suppose that it is required to find a function $u(M, t)$ satisfying the equation

$$\operatorname{div}(k\,\nabla u)-qu=\begin{cases}\rho\, u_{tt}\\ \rho\, u_t\end{cases} \tag{1}$$

for $t>0$ in the domain $D$ bounded by a closed piecewise-smooth surface $S$ which is continuous in the closed domain $\bar{B}\equiv\{M\in\bar{D};\ t\geqslant 0\}$, where $\bar{D}=D+S$, subject to the additional conditions

$$\left(\gamma_1\frac{\partial u}{\partial n}+\gamma_2 u\right)_S=0 \tag{2}$$

and

$$u(M,0)=\varphi(M),\qquad u_t(M,0)=\varphi_1(M)$$
$$[\text{respectively } u(M,0)=\varphi(M)] \tag{3}$$

Substituting $L[u] \equiv \operatorname{div}(k\nabla u) - qu$, Equation (1) can be rewritten in the form

$$L[u] = \begin{cases} \rho u_{tt} \\ \rho u_t \end{cases} \tag{1'}$$

This equation and the boundary conditions given by (2) are linear and homogeneous. Consequently, if $u_1$ and $u_2$ are the solutions of (1), which satisfy the condition given by (2), then the functions

$$u = c_1 u_1 + c_2 u_2$$

where $c_1$ and $c_2$ are constants will also be solutions of (1) subject to the conditions (2).

We shall try to satisfy the initial conditions (3) by superimposing all the linearly independent initial solutions of this type (i.e. solutions satisfying the boundary conditions (2)). We shall thus seek non-trivial solutions of Equation (1), i.e. those which are not identically zero, which satisfy the boundary conditions (2) among functions of the form $\Phi(M)\,\Psi(t)$, where $\Phi(M)$ is continuous in $\bar{D}$ and $\Psi(t)$ is continuous for $0 \leqslant t < \infty$. Substituting the function $\Phi(M)\,\Psi(t)$ into (1) and dividing both sides of the equation by $\rho(M)\,\Phi(M)\,\Psi(t)$, we obtain

$$\frac{L[\Phi]}{\rho\Phi} = \frac{\Psi''}{\Psi} \qquad \left(\text{respectively } \frac{\Psi'}{\Psi}\right)$$

In order that this equation should be an identity, i.e. the function $\Phi(M)\,\Psi(t)$ should satisfy (1), it is necessary and sufficient that both fractions $L[\Phi]/\rho\Phi$ and $\Psi''/\Psi$ should be equal to the same constant, i.e.

$$\frac{L[\Phi]}{\rho\Phi} = -\lambda = \frac{\Psi''}{\Psi}$$

The following identities must therefore be satisfied

$$\Psi'' + \lambda\Psi \equiv 0 \qquad (\Psi' + \lambda\Psi \equiv 0) \quad \text{and} \quad L[\Phi] + \lambda\rho\Phi \equiv 0$$

Consequently, we can take the functions $\Psi(t)$ and $\Phi(M)$ to be the non-trivial solutions of the equations

$$\Psi'' + \lambda\Psi = 0 \quad (\text{respectively } \Psi' + \lambda\Psi = 0) \tag{4}$$

$$L[\Phi] + \lambda\rho\Phi = 0 \tag{5}$$

where $\Phi(M)$ must satisfy the boundary condition

$$\left(\gamma_1 \frac{\partial\Phi}{\partial n} + \gamma_2\Phi\right)_S = 0 \tag{6}$$

The problem defined by (5)–(6) is called the Sturm–Liouville problem. It does not possess non-trivial solutions for all values of $\lambda$.

*Definition* Those values of $\lambda$ for which the problem (5)–(6) has non-trivial solutions are called the eigenvalues of the boundary-value problem (5)–(6), and the corresponding non-trivial solutions $\Phi(M)$ of (5) are called the eigenfunctions of the boundary-value problem (5)–(6).

We shall assume henceforth that $k(M), q(M)$ and $\rho(M)$ are continuous in $\bar{D}$, $k(M) > 0, \rho(M) \geqslant 0$ (but $\rho(M) \not\equiv 0$), $q(M) \geqslant 0$ in $\bar{D}$; $\gamma_1(M)$ and $\gamma_2(M) \geqslant 0$ on $S$ and $\gamma_1^2+\gamma_2^2 \neq 0$.

Subject to these conditions we have the following theorem.

*Theorem 1* There exists an infinite set of eigenvalues $\{\lambda_n\}$ ($n = 1, 2, \ldots$) and corresponding eigenfunctions $\{\Phi_n(M)\}$ of the boundary-value problem (5)–(6).

We shall not reproduce the proof of this theorem here.

We shall say that the function $f(M)$ has piecewise-continuous first- and second-order partial derivatives in $\bar{D}$ if there exists a finite number of domains $D_1, D_2, \ldots, D_n$ which do not intersect in pairs and are such that (1) $\bar{D} = \bar{D}_1+\bar{D}_2+ \ldots +\bar{D}_n$ and (2) $f(M)$ has continuous and bounded first- and second-order partial derivatives in each of the domains $D_1, D_2, \ldots, D_n$.

Let $A$ denote the class of functions which (1) are continuous in $\bar{D}$, (2) have piecewise-continuous first- and second-order partial derivatives in $\bar{D}$ and (3) satisfy the boundary conditions (6).

The eigenfunctions of the boundary-value problem (5)–(6) will clearly belong to class $A$ but they do not exhaust all the possible members of this class of functions.

**4.1.2** The eigenvalues and eigenfunctions of the boundary-value problem (5)–(6) have a number of properties and we shall now summarise some of them.

*Expansion theorem* Any function $f(M)$ belonging to class $A$ can be expanded into a Fourier series in terms of the eigenfunctions of the boundary-value problem (5)–(6), and the expansion will converge absolutely and uniformly in the domain $\bar{D}$.

We shall not reproduce the proof of this theorem here (the one-dimensional case will be discussed in Chapter 9).

The Fourier series for the function $f(M)$ in terms of the functions $\{\Phi_n(M)\}$ is defined as the series $\sum_{n=1}^{\infty} C_n\Phi_n(M)$, in which the coefficients $C_n$ are given by

$$C_n = \frac{1}{\int\limits_D \rho\Phi_n^2 \, d\tau} \int\limits_D f(M)\rho(M)\Phi_n(M) \, d\tau$$

Having solved the Sturm–Liouville problem, let us return to Equation (4). For each eigenvalue $\lambda_n$ we have the general solution

$$\Psi_n(t) = C_n \cos\sqrt{\lambda_n}t + D_n \sin\sqrt{\lambda_n}t$$
$$(\text{where } \Psi_n(t) = C_n\, e^{-\lambda_n t})$$

Therefore, partial solutions of (1) which satisfy only the boundary conditions (2) are functions of the form

$$u_n(M, t) = (C_n \cos\sqrt{\lambda_n}t + D_n \sin\sqrt{\lambda_n}t)\Phi_n(M)$$
$$(\text{where } u_n(M, t) = C_n\, e^{-\lambda_n t}\Phi_n(M))$$

These functions can be written in the form

$$u_n = B_n \sin(\sqrt{\lambda_n}t + \theta_n)\Phi_n(M)$$

where $B_n = \sqrt{C_n^2 + D_n^2}$, $\theta_n = \tan^{-1}(C_n/D_n)$.

Functions of this kind are said to describe natural vibrations and standing waves; $u_1(M, t)$ is the fundamental and $u_2(M, t)$, $u_3(M, t)$ are the harmonics. The numbers $\sqrt{\lambda_1}$, $\sqrt{\lambda_2}$ are the natural frequencies of the vibrations. These frequencies do not depend on the initial conditions. This means that the frequencies of natural vibrations are independent of the method used to excite them. They characterise the properties of the vibrating system itself and are determined by the material constants of the system (for example, the velocity of sound in the medium), geometrical factors (shape and dimensions) and the conditions on the boundary.

The eigenfunction $B_n\Phi_n(M)$ specifies the profile of the standing wave.

If we now take the sum of these special solutions over all the eigenfunctions

$$u(M, t) = \sum_{n=1}^{\infty} (C_n \cos\sqrt{\lambda_n}t + D_n \sin\sqrt{\lambda_n}t)\Phi_n(M) \qquad (7)$$

we encounter the following problem: is it possible to choose the coefficients $C_n$ and $D_n$ so that the sum will be the solution of the problem (1)–(3)? The answer to this problem is given by the following theorem.

*Theorem 2* The solution of (1)–(3) in a closed domain $\bar{B} \equiv \{M \in \bar{D};\ t \geqslant 0\}$ can be represented by the series (7) with

$$C_n = \frac{1}{\|\Phi_n\|^2} \int_D \rho(P)\,\varphi(P)\,\Phi_n(P)\,\mathrm{d}\tau_P$$

$$D_n = \frac{1}{\|\Phi_n\|^2} \int_D \rho(P)\,\varphi_1(P)\,\Phi_n(P)\,\mathrm{d}\tau_P$$

$$\|\Phi_n\|^2 = \int_D \rho(P)\,\Phi_n^2(P)\,\mathrm{d}\tau_P$$

The number $\|\Phi_n\|$ is called the norm of the function $\Phi_n(M)$.

Consider the functional

$$R[w, \Phi] = -\int_D wL[\Phi]\,\mathrm{d}\tau_P$$

The following lemma applies to this expression.

*Lemma* The functional $R[w, \Phi]$ is symmetric for the class $A$, i.e. $R[w, \Phi] = R[\Phi, w]$, for any functions $w$ and $\Phi$ belonging to class $A$.

*Proof* Using the well-known result

$$p\,\mathrm{div}(\boldsymbol{E}) = \mathrm{div}(p\boldsymbol{E}) - \boldsymbol{E}\nabla p, \qquad p = w, \qquad \boldsymbol{E} = k\nabla\Phi$$

we have (see definition of $L$ in Section 4.1.1).

$$R[w, \Phi] = \int_D k(\nabla w \cdot \nabla\Phi)\,\mathrm{d}\tau + \int_D qw\Phi\,\mathrm{d}\tau - \int_D \mathrm{div}(k \cdot w \cdot \nabla\Phi)\,\mathrm{d}\tau$$

In view of Green's theorem, the last integral is equal to $\int_S kw\frac{\partial\Phi}{\partial n}\,\mathrm{d}\tau$, and therefore

$$R[w, \Phi] = \int_D k(\nabla w \cdot \nabla\Phi)\,\mathrm{d}\tau + \int_D qw\Phi\,\mathrm{d}\tau - \int_S kw\frac{\partial\Phi}{\partial n}\,\mathrm{d}\sigma \tag{8}$$

If we are dealing with the first or second boundary-value problem, we have $\int_S kw\frac{\partial\Phi}{\partial n}\,\mathrm{d}\tau = 0$, so that $w\Big|_S = 0 \left(\frac{\partial\Phi}{\partial n}\Big|_S = 0\right)$. In all

these cases $R[w, \Phi]$ must be symmetric. In the case of the third boundary-value problem, we have from the boundary conditions $\left.\frac{\partial \Phi}{\partial n}\right|_S = -\left.\frac{\gamma_2}{\gamma_1}\Phi\right|_S$, and therefore $R[w, \Phi]$ can be written in the form

$$R[w, \Phi] = \int_D k(\nabla w \cdot \nabla \Phi)\, d\tau + \int_D qw\Phi\, d\tau + \int_S \frac{\gamma_2}{\gamma_1} kw\Phi\, d\sigma$$

This expression is again symmetric.

*Remark* For functions belonging to class $A$ we must have $R[\Phi, \Phi] \geqslant 0$.

*Proof of Theorem 2* Suppose that $u(M, t)$ is the required solution. Since this solution is continuous in $\bar{D}$ and satisfies the boundary conditions (2) for any $t > 0$, it must belong to class $A$. Consequently, in view of the expansion theorem it can be represented by the Fourier series

$$u(M, t) = \sum_{n=1}^{\infty} \Psi_n(t)\, \Phi_n(M) \tag{9}$$

where

$$\Psi_n(t) = \frac{1}{\|\Phi_n\|^2} \int_D \rho(P)\, u(P, t)\, \Phi_n(P)\, d\tau_P \tag{10}$$

Using Equation (5) for $\Phi_n(M)$ we can transform the last formula to the form

$$\Psi_n(t) = \frac{-1}{\lambda_n \|\Phi_n\|^2} \int_D uL[\Phi_n]\, d\tau = \frac{R[u, \Phi_n]}{\lambda_n \|\Phi_n\|^2}$$

and, using the above lemma,

$$\Psi_n(t) = \frac{R[\Phi_n, u]}{\lambda_n \|\Phi_n\|^2} = \frac{-1}{\lambda_n \|\Phi_n\|^2} \int_D \Phi_n(P) L[u]\, d\tau$$

Using (1) we obtain

$$\Psi_n(t) = \frac{-1}{\lambda_n \|\Phi_n\|^2} \int_D \rho\, \Phi_n u_{tt}\, d\tau$$

and hence, if we compare this result with (10) we have

$$\Psi_n(t) \equiv \frac{-\Psi_n''}{\lambda_n}, \quad \text{i.e.} \quad \Psi_n'' + \lambda_n \Psi \equiv 0$$

Therefore the function $\Psi_n(t)$ is a solution of the equation $\Psi'' + \lambda_n \Psi = 0$ and, consequently, can be written in the form

$$\Psi_n(t) = C_n \cos \sqrt{\lambda_n}\, t + D_n \sin \sqrt{\lambda_n}\, t$$

where

$$C_n = \Psi_n(0), \qquad D_n \sqrt{\lambda_n} = \Psi_n'(0)$$

Using (10), we find that

$$C_n = \Psi_n(0) = \frac{1}{\|\Phi_n\|^2} \int_D \rho(P) u(P, 0) \Phi_n(P)\, d\tau$$

$$= \frac{1}{\|\Phi_n\|^2} \int_D \rho(P) \varphi(P) \Phi_n(P)\, d\tau$$

$$D_n = \frac{\Psi_n'(0)}{\sqrt{\lambda_n}} = \frac{1}{\sqrt{\lambda_n}\|\Phi_n\|^2} \int_D \rho(P) \varphi_1(P) \Phi_n(P)\, d\tau$$

which was to be established.

By assuming the existence of the solution of (1)–(3), we have come to the conclusion that it can be represented by the series (7) and, therefore, it is a unique solution.

**4.1.3** Let us now consider the following properties of eigenfunctions and eigenvalues.

*Property 1* If $\Phi$ is an eigenfunction corresponding to an eigenvalue $\lambda$, then $C\Phi$ (where $C$ is a constant) is also an eigenfunction corresponding to the same eigenvalue.

*Property 2* If $\Phi_1$ and $\Phi_2$ are eigenfunctions corresponding to the eigenvalue $\lambda$, any linear combination such as $C_1\Phi_1 + C_2\Phi_2$ is also an eigenfunction corresponding to the same eigenvalue $\lambda$. The validity of these statements is obvious.

*Property 3* Eigenfunctions $\Phi_1$ and $\Phi_2$ corresponding to different eigenvalues $\lambda_1$ and $\lambda_2$ ($\lambda_1 \neq \lambda_2$) are orthogonal in the domain $D$ with a weight $\rho(M)$, i.e.

$$\int_D \rho(P) \Phi_1(P) \Phi_2(P)\, d\tau = 0$$

*Proof* By definition of eigenfunctions and eigenvalues we have

$$L[\Phi_1]+\lambda_1\rho\Phi_1 \equiv 0$$
$$L[\Phi_2]+\lambda_2\rho\Phi_2 \equiv 0$$

Multiplying the first of these by $\Phi_2$ and the second by $\Phi_1$, subtracting one from the other and integrating the resulting identity over the domain $D$, we obtain

$$-R[\Phi_2, \Phi_1]+R[\Phi_1, \Phi_2] = (\lambda_2-\lambda_1)\int_D \rho\,\Phi_1\Phi_2\,\mathrm{d}\tau$$

Since the functional $R$ is symmetric (the eigenfunction belongs to class $A$), the left-hand side of this equation is zero and, consequently, $\int_D \rho\Phi_1\Phi_2\mathrm{d}\tau = 0$, since $\lambda_1 \neq \lambda_2$, which was to be established.

If to a given eigenvalue $\lambda$ there correspond $r$ linearly independent eigenfunctions $\Phi_1, \Phi_2, \ldots, \Phi_r$, these functions will not necessarily be orthogonal. It is possible, however, to replace them by other eigenfunctions $\tilde{\Phi}_1, \tilde{\Phi}_2, \ldots, \tilde{\Phi}_r$, which are linear combinations of the original eigenfunctions and these can be made to be orthogonal. In fact, let us suppose that $\tilde{\Phi}_1 = \Phi_1$. If $\int_D \rho\Phi_1\Phi_2\,\mathrm{d}\tau = 0$, then $\tilde{\Phi}_2 = \Phi_2$ and if $\int_D \rho\Phi_1\Phi_2\mathrm{d}\tau \neq 0$, then $\tilde{\Phi}_2 = \tilde{\Phi}_1+B_1\Phi_2$. The constant $B_1$ can be found from the condition $\int_D \rho\tilde{\Phi}_1\tilde{\Phi}_2\mathrm{d}\tau = 0$, i.e. from the equation

$$\int_D \rho\tilde{\Phi}_1^2\,\mathrm{d}\tau+B_1\int_D \rho\tilde{\Phi}_1\tilde{\Phi}_2\,\mathrm{d}\tau = 0$$

If $\tilde{\Phi}_3$ are orthogonal to $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$, then we can set $\tilde{\Phi}_3 = \Phi_3$ and, if they are not, we can set $\tilde{\Phi}_3 = \tilde{\Phi}_1+B_{32}\tilde{\Phi}_2+B_{33}\Phi_3$. The constants $B_{32}$ and $B_{33}$ can be found from the conditions $\int_D \rho\tilde{\Phi}_1\tilde{\Phi}_3\,\mathrm{d}\tau = 0$ and $\int_D \rho\tilde{\Phi}_2\tilde{\Phi}_3\mathrm{d}\tau = 0$, i.e. from the equations

$$\int_D \rho\tilde{\Phi}_1^2\,\mathrm{d}\tau+B_{32}\int_D \rho\tilde{\Phi}_1\tilde{\Phi}_2\,\mathrm{d}\tau+B_{33}\int_D \rho\tilde{\Phi}_1\Phi_3\mathrm{d}\tau = 0$$

$$\int_D \rho\tilde{\Phi}_1\tilde{\Phi}_2\,\mathrm{d}\tau+B_{32}\int_D \rho\tilde{\Phi}_2^2\,\mathrm{d}\tau+B_{33}\int_D \rho\tilde{\Phi}_2\Phi_3\mathrm{d}\tau = 0$$

and so on.

Continuing this orthogonalisation process, we can consider $r$ eigenfunctions $\tilde{\Phi}_1, \tilde{\Phi}_2, \ldots, \tilde{\Phi}_r$ corresponding to the given eigenvalue $\lambda$ and these will now be orthogonal in pairs. Assuming further that this orthogonalisation process has been performed, we conclude that any two linearly independent eigenfunctions of the boundary value problem (5)–(6) are orthogonal in the domain $D$ with a weight $\rho$.

*Property 4* All the eigenvalues of the problem (5)–(6) are real.

Suppose that $\lambda = \alpha + i\beta$ ($\beta \neq 0$) is an eigenvalue and $\Phi = \Phi_1 + i\Phi_2$ is the corresponding eigenfunction. We must then have

$$L[\Phi_1 + i\Phi_2] + (\alpha + i\beta)\rho(\Phi_1 + i\Phi_2) \equiv 0$$

and, consequently,

$$L[\Phi_1] + \alpha\rho\Phi_1 - \beta\rho\Phi_2 \equiv 0$$
$$i\{L[\Phi_2] + \alpha\rho\Phi_2 + \beta\rho\Phi_1\} \equiv 0$$

Subtracting the second identity from the first, we obtain

$$L[\Phi_1 - i\Phi_2] + (\alpha - i\beta)\rho(\Phi_1 - i\Phi_2) \equiv 0$$

Therefore, $\bar{\lambda} = \alpha - i\beta$ and $\bar{\Phi} = \Phi_1 - i\Phi_2$ are the eigenvalue and eigenfunction of the problem, respectively. Using Property 3 we have

$$\int_D \rho(\Phi_1 + i\Phi_2)(\Phi_1 - i\Phi_2)\, d\tau = 0, \quad \text{where} \quad \int_D \rho(\Phi_1^2 + \Phi_2^2)\, d\tau = 0$$

which is impossible.

*Property 5* All the eigenvalues of the problem (5)–(6) are non-negative.

To prove this, we multiply the identity $L[\Phi_n] + \lambda_n\rho\Phi_n \equiv 0$ by $\Phi_n$ and integrate the result over the domain $D$. The result is

$$\int_D \Phi_n L[\Phi_n]\, d\tau + \lambda_n \int_D \rho\Phi_n^2\, d\tau = 0$$

and hence

$$\lambda_n = \frac{R[\Phi_n, \Phi_n]}{\|\Phi_n\|^2}$$

Since $R[\Phi_n, \Phi_n] \geqslant 0$, we have $\lambda_n \geqslant 0$.

*Remark* For the first and third boundary-value problems all the eigenvalues are positive. For the second eigenvalue problem with $q(M) \equiv 0$, we have $\lambda = 0$ as the eigenvalue and $\Phi \equiv 1$ as the corresponding eigenfunction.

*Example 1* Suppose that it is required to solve the problem

$$a^2 u_{xx} = u_{tt}$$
$$u(x, 0) = \varphi(x), \qquad u_t(x, 0) = \varphi_1(x), \qquad u(0, t) = u(l, t) = 0 \quad (11)$$
$$\varphi(0) = \varphi(l) = 0$$

We recall that we solved this problem in Chapter 3 by the method of characteristics. We then continued the initial values $\varphi(x)$ and $\varphi_1(x)$ to the segment $(-l, 0)$, as odd functions and then periodically over the whole straight line. We leave it to the reader to show directly that the solution obtained by the method of characteristics is the same as that obtained by the method of separation of variables.

*Solution* Among the functions of the form $\Phi(x)\Psi(t)$ there are solutions of (11) which satisfy only the boundary conditions of the problem. Substituting $\Phi(x)\Psi(t)$ into the equation, we obtain

$$\frac{\Phi''}{\Phi} \equiv \frac{\Psi''}{a^2\Psi} = -\lambda$$

Consequently, $\Psi'' + a^2\lambda\Psi = 0$ and

$$\Phi'' + \lambda\Phi = 0, \qquad \Phi(0) = \Phi(l) = 0 \qquad (12)$$

This is the first boundary-value problem. All its eigenvalues are positive. The general solution of (12) can therefore be written in the form

$$\Phi(x) = A \cos\sqrt{\lambda}\, x + B \sin\sqrt{\lambda}\, x$$

From the boundary condition on the left we find that $A = 0$. Consequently, $\Phi(x) = B \sin\sqrt{\lambda}\, x$ and $B \neq 0$. From the boundary conditions on the right we have $B \sin\sqrt{\lambda}\, l = 0$. Consequently, $\sin\sqrt{\lambda}\, l = 0$ and hence $\sqrt{\lambda}\, l = n\pi$ and

$$\lambda_n = \frac{n^2\pi^2}{l^2} \qquad (n = 1, 2, 3, \ldots)$$

These are the eigenvalues. The corresponding eigenfunctions are $\Phi_n(x) = \sin(n\pi/l)x$.

For each $\lambda_n$ we have

$$\Psi_n(t) = C_n \cos\frac{a\pi n}{l}t + D_n \sin\frac{a\pi n}{l}t$$

According to Theorem 2 (Section 4.1.2), the required solution of the problem is the function

$$u(x, t) = \sum_{n=1}^{\infty}\left(C_n \cos\frac{a\pi n}{l}t + D_n \sin\frac{a\pi n}{l}t\right)\sin\frac{\pi n}{l}x$$

where

$$C_n = \frac{2}{l}\int_0^l \varphi(\xi)\sin\frac{\pi n}{l}\xi\,d\xi, \qquad D_n = \frac{2}{a\pi n}\int_0^l \varphi_1(\xi)\sin\frac{\pi n}{l}\xi\,d\xi$$

$$\|\Phi_n\|^2 = \int_0^l \sin^2\frac{\pi n}{l}\xi\,d\xi = \frac{l}{2}$$

*Example 2* It is required to solve the problem

$$a^2u_{xx} = u_t, \qquad u(x, 0) = \varphi(x), \qquad u_x(0, t) = u_x(l, t) = 0$$

As in the preceding example, we find that

$$\Phi'' + \lambda\Phi = 0, \qquad \Phi'(0) = \Phi'(l) = 0, \qquad \Psi' + a^2\lambda\Psi = 0 \quad (13)$$

We thus have the second boundary-value problem $q \equiv 0$. Consequently, $\lambda = 0$ is an eigenvalue and $\Phi(x) \equiv l$ is the corresponding eigenfunction.

The remaining eigenvalues and eigenfunctions can be found as in Example 1:

$$\Phi(x) = A\cos\sqrt{\lambda}\,x + B\sin\sqrt{\lambda}\,x$$

From the condition $\Phi'(0) = 0$ we find that $B = 0$. Consequently, $A \neq 0$ and $\Phi(x) = A\cos\sqrt{\lambda}x$. From the condition $\Phi'(l) = 0$ we find that $\sin\sqrt{\lambda}\,l = 0$ and consequently $\sqrt{\lambda}\,l = \pi n$ and $\lambda_n = \pi^2n^2/l^2$ $(n = 1, 2, 3, \ldots)$. It follows that

$$0, \frac{\pi^2}{l^2}, \frac{4\pi^2}{l^2}, \ldots, \frac{\pi^2n^2}{l^2}, \ldots$$

are the eigenvalues and

$$1,\ \cos\frac{\pi}{l}x,\ \cos\frac{2\pi}{l}x,\ \ldots,\ \cos\frac{\pi n}{l}x,\ \ldots$$

are the eigenfunctions. For each $\lambda_n$ we can find the following eigenfunctions

$$\Psi_n(t) = C_n\,\mathrm{e}^{-a^2\lambda_n t} \qquad (n = 0, 1, 2, \ldots)$$

The required solution of the problem will, in view of Theorem 2 (Section 4.1.2) be the function

$$u(x, t) = \sum_{n=0}^{\infty} C_n\,\mathrm{e}^{-a^2\lambda_n t}\cos\frac{\pi n}{l}x$$

where

$$C_0 = \frac{1}{l}\int_0^l \varphi(\xi)\,\mathrm{d}\xi, \quad C_n = \frac{2}{l}\int_0^l \varphi(\xi)\cos\frac{\pi n}{l}\xi\,\mathrm{d}\xi \quad (n = 1, 2, 3\ \ldots)$$

$$\|\Phi_0\|^2 = l, \qquad \|\Phi_n\|^2 = \int_0^l \cos^2\frac{\pi n}{l}\xi\,\mathrm{d}\xi = \frac{l}{2} \quad (n = 1, 2, \ldots)$$

*Example 3* It is required to determine the temperature distribution along a homogeneous rod of length $l$ whose lateral surface is insulated and is subject to convective heat transfer (Newton's law) at the ends. The surrounding media are at constant temperatures $u_1$ and $u_2$, respectively. The initial temperature is arbitrary.

The mathematical formulation of the problem is as follows:

$$a^2 u_{xx} = u_t \tag{14}$$

$$u_x(0, t) - h_1[u(0, t) - u_1] = 0 \tag{15}$$

$$u_x(l, t) + h_2[u(l, t) - u_2] = 0 \tag{16}$$

$$u(x, 0) = \varphi(x) \tag{17}$$

We shall seek the solution in the form $u(x, t) = v(x) + w(x, t)$, where $v(x)$ is the solution of (14) satisfying the boundary conditions (15) and (16), i.e.

$$v'' = 0 \tag{$14_1$}$$

$$v'(0) - h_1[v(0) - u_1] = 0 \tag{$15_1$}$$

$$v'(l) + h_2[v(l) - u_2] = 0 \tag{$16_1$}$$

For the functions $w(x, t)$ the problem may be formulated as follows:

$$a^2 w_{xx} = w_t \tag{$14_2$}$$

$$w_x(0, t) - h_1 w(0, t) = 0 \tag{$15_2$}$$

$$w_x(l, t) + h_2 w(l, t) = 0 \tag{$16_2$}$$

$$w(x, 0) = \varphi_1(x) = \varphi(x) - v(x) \tag{$17_2$}$$

The function $v(x)$ describes the stationary state and $w(x, t)$ a departure from the state.

Let us solve the problem for $v(x)$ first. The general solution of $(14_1)$ is of the form

$$v(x) = C_1 x + C_2$$

The constants $C_1$ and $C_2$ are given by $(15_1)$ and $(16_1)$:

$$C_1 - h_1(C_2 - u_1) = 0$$

$$C_1 + h_2(C_1 l + C_2 - u_2) = 0$$

and hence

$$C_1 = \frac{h_1(u_2 - u_1)}{h_1 + h_2 + h_1 h_2 l}, \qquad C_2 = u_1 + \frac{C_1}{h_1}$$

We have thus found the steady-state solution. The problem for $w(x, t)$ will now be solved by the method of separation of variables. Among the functions of the form $\Phi(x)\Psi(t)$ we shall require the solutions of $(14_2)$ which satisfy only the boundary conditions $(15_2)$ and $(16_2)$. Substituting the function $\Phi(x)\Psi(t)$ into $(14_2)$ and into the boundary conditions $(15_2)$ and $(16_2)$, we obtain

$$\Phi'' + \lambda\Phi = 0 \tag{18}$$

$$\Phi'(0) - h_1\Phi(0) = 0 \tag{19}$$

$$\Phi'(l) + h_2\Phi(l) = 0 \tag{20}$$

$$\Psi' + a^2\lambda\Psi = 0 \tag{21}$$

In view of Property 5, the problem defined by (18)–(20) will have only positive eigenvalues. Therefore, the general solution of (18) can be written in the form

$$\Phi(x) = A\cos\sqrt{\lambda}\,x + B\sin\sqrt{\lambda}\,x$$

From the boundary condition (19) we find that $B\sqrt{\lambda} = h_1 A$ and consequently

$$\Phi(x) = \frac{B}{h_1}\left(\sqrt{\lambda}\cos\sqrt{\lambda}\,x + h_1\sin\sqrt{\lambda}\,x\right) \tag{22}$$

The factor $B(1/h_1)$ will be included in $\Psi(t)$. Substituting (22) into (20), we obtain the equation for the eigenvalues:

$$\cos\mu = \frac{1}{l(h_1+h_2)}\left(\mu - \frac{h_1 h_2 l^2}{\mu}\right)$$

where $\mu = \sqrt{\lambda}\, l$. $\mu_1, \mu_2, \ldots, \mu_n, \ldots$ are the positive roots of this equation. The eigenvalues will then be the numbers

$$\lambda_n = \frac{\mu_n^2}{l^2}$$

The eigenfunctions will be of the form

$$\Phi_n(x) = \frac{\mu_n}{l}\cos\frac{\mu_n}{l}x + h_1 \sin\frac{\mu_n}{l}x$$

They are orthogonal within the range $[0, l]$ with a weight $\rho \equiv 1$.

Let us return now to Equation (21). Its general solution for $\lambda = \lambda_n$ is of the form

$$\Psi_n(t) = C_n\, \mathrm{e}^{-a^2\lambda_n t}$$

We then have

$$w(x, t) = \sum_{n=1}^{\infty} C_n\, \mathrm{e}^{-a^2\lambda_n t}\, \Phi_n(x)$$

The coefficients $C_n$ can be found from the initial condition using the orthogonality of the eigenfunctions $\Phi_n(x)$:

$$C_n = \frac{1}{\|\Phi_n\|^2}\int_0^l \varphi_1(\xi)\left(\frac{\mu_n}{l}\cos\frac{\mu_n}{l}\xi + h_1 \sin\frac{\mu_n}{l}\xi\right)\mathrm{d}\xi$$

$$\|\Phi_n\|^2 = \int_0^l \left(\frac{\mu_n}{l}\cos\frac{\mu_n}{l}\xi + h_1 \sin\frac{\mu_n}{l}\xi\right)^2 \mathrm{d}\xi$$

*Remark* The method discussed in this example can be used for a broad class of boundary value problems with steady-state, i.e. time-independent, inhomogeneous parts, either in the equation itself or in the boundary conditions (or both).

*Example 4* Determine the transverse vibrations of a string, one end of which is rigidly fixed and the other free, if a localised mass $m_0$ is attached to the free end and the initial excitation is arbitrary.

The mathematical formulation of the problem is as follows:

$$a^2 u_{xx} = u_{tt}, \qquad a^2 = \frac{T}{\rho_0}$$

$$u(x, 0) = \varphi(x), \qquad u_t(x, 0) = \varphi_1(x)$$

$$u(0, t) = 0, \qquad Tu_x(l, t) = m_0 u_{tt}(l, t)$$

Among the class of functions $\Phi(x)\Psi(t)$ we shall seek solutions which satisfy only the boundary conditions. Separating the variables we have

$$\Psi'' + a^2\lambda\Psi = 0 \tag{23}$$

$$\Phi'' + \lambda\Phi = 0 \tag{24}$$

$$\Phi(0) = 0 \tag{25}$$

The boundary condition on the right can be written in the form

$$T\Phi'(l)\Psi(t) - m_0\Phi(l)\Psi''(t) = 0$$

Replacing $\Psi''(t)$ in (23) by $\Psi(t)$ and dividing both sides of the equation by $\Psi(t)$, we obtain

$$\Phi'(l) + h\lambda\Phi(l) = 0 \tag{26}$$

where $h = a^2 m_0/T$.

The solutions of (24) which satisfy (25) are of the form $\Phi(x) = \sin\sqrt{\lambda}x$. From (26) we find the equation for the eigenvalues $\lambda_n > 0$:

$$\tan\mu = \frac{-1}{h\mu}, \qquad \mu = \sqrt{\lambda}\, l$$

The corresponding eigenfunctions are

$$\Phi_n(x) = \sin\frac{\mu_n}{l}x$$

It is readily verified directly that these functions are not orthogonal with the weight $\rho(x) \equiv 1$. This does not conflict with the general theorem on the orthogonality of eigenfunctions since the boundary condition (3) is not the usual boundary condition of type III because it contains explicitly (rather than through the eigenfunction) the eigenvalue $\lambda$. To elucidate this point we note that the equation for $u(x, t)$ can be written in the form

$$Tu_{xx} = [\rho_0 + m_0\delta(x-l)]\, u_{tt}$$

Consequently, the equation for the eigenfunctions can be written in the form

$$T\Phi''+\lambda\rho(x)\Phi = 0$$

where

$$\rho(x) = \rho_0+m_0\delta(x-l)$$

The eigenfunctions $\Phi_n(x)$ will therefore be orthogonal with the weight $\rho(x)$. This is readily verified by direct calculation.

The next step is the same as before. First, we find $\Psi_n(t)$ so that

$$u(x, t) = \sum_{n=1}^{\infty}\left(C_n\cos\frac{a\mu_n}{l}t+D_n\sin\frac{a\mu_n}{l}t\right)\sin\frac{\mu_n}{l}x$$

From the initial conditions we can determine the coefficients $C_n$ and $D_n$ using the orthogonality of the eigenfunctions in terms of the weight $\rho = \rho_0+m_0\delta(x-l)$:

$$C_n = \frac{1}{\|\Phi_n\|^2}\left\{\int_0^l \rho_0\varphi(\xi)\,\Phi_n(\xi)\,\mathrm{d}\xi+m_0\varphi(l)\,\Phi_n(l)\right\}$$

$$D_n = \frac{l}{a\mu_n\|\Phi_n\|^2}\left\{\int_0^l \rho_0\varphi_1(\xi)\,\Phi_n(\xi)\,\mathrm{d}\xi+m_0\varphi_1(l)\,\Phi_n(l)\right\}$$

$$\|\Phi_n\|^2 = \rho_0\int_0^l \Phi_n^2(\xi)\,\mathrm{d}\xi+m_0\Phi_n^2(l)$$

Let us now return to the properties of the eigenvalues and eigenfunctions. We note, to begin with, that since for the functions $\Phi$ belonging to class $A$ we have $R[\Phi, \Phi] \geqslant 0$,

$$\inf_{\Phi\in A}\frac{R[\Phi, \Phi]}{\|\Phi\|^2} = \mu \geqslant 0$$

*Property 6 (extremal property)* This is expressed by the following theorem.

*Theorem 3* If $\mu = \inf\limits_{\Phi\in A}\dfrac{R[\Phi, \Phi]}{\|\Phi\|^2}$ is valid for some function $\tilde{\Phi}$ belonging to class $A$, then $\tilde{\Phi}$ is an eigenfunction and $\mu$ is the corresponding eigenvalue for the problem (5)–(6). At the same time, $\mu$ will, of course, be the smallest eigenvalue.

*Proof* For any function $\Phi$ belonging to class $A$ we have $\dfrac{R[\Phi, \Phi]}{\|\Phi\|^2} - \mu \geqslant 0$. In particular, $\lambda_n = \dfrac{R[\Phi_n, \Phi_n]}{\|\Phi_n\|^2} \geqslant \mu$. Consequently, for $\Phi \in A$, we have

$$\Psi[\Phi] \equiv R[\Phi, \Phi] - \mu \|\Phi\|^2 \geqslant 0$$

whereas

$$\Psi[\tilde{\Phi}] = R[\tilde{\Phi}, \tilde{\Phi}] - \mu \|\tilde{\Phi}\|^2 = 0$$

Therefore, the functional $\Phi[\Psi]$ reaches a minimum on $\tilde{\Phi}$. This is equivalent to $\varphi(\alpha) = \Psi[\tilde{\Phi}+\alpha f]$, where $f \in A$ reaches a minimum for $\alpha = 0$. However, we then have $\varphi'(0) = 0$. Let us evaluate this derivative:

$$\begin{aligned}
\varphi'(0) &= \frac{\mathrm{d}}{\mathrm{d}\alpha}\{R[\tilde{\Phi}+\alpha f, \tilde{\Phi}+\alpha f] - \mu \|\tilde{\Phi}+\alpha f\|^2\}_{\alpha=0} \\
&= -\frac{\mathrm{d}}{\mathrm{d}\alpha}\left\{\int_D (\tilde{\Phi}+\alpha f) L[\tilde{\Phi}+\alpha f]\, \mathrm{d}\tau - \mu \int_D \rho (\tilde{\Phi}+\alpha f)^2\, \mathrm{d}\tau\right\}_{\alpha=0} \\
&= -\int_D \{f L[\tilde{\Phi}] + \Phi L[f]\}\, \mathrm{d}\tau - 2\mu \int_D \rho\, \tilde{\Phi} f\, \mathrm{d}\tau \\
&= -2 \int_D f\{L[\tilde{\Phi}] + \mu\rho\tilde{\Phi}\}\, \mathrm{d}\tau
\end{aligned}$$

where we have used the symmetry property of the functional $R[f, \tilde{\Phi}]$ for functions belonging to class $A$.

Therefore, for an arbitrary function $f$ belonging to $A$ we have

$$\int_D f\{L[\tilde{\Phi}] + \mu\rho\tilde{\Phi}\}\, \mathrm{d}\tau = 0$$

Hence, it follows (from the fundamental lemma of the calculus of variations) that at points at which the function $L[\tilde{\Phi}]$ is continuous we must have

$$L[\tilde{\Phi}] + \mu\rho\tilde{\Phi} \equiv 0$$

which was to be established.

If the minimum of the functional is sought in a class of functions $A_k$ belonging to $A$ and orthogonal with respect to the weight $\rho$ in the domain $D$ to the eigenfunctions $\Phi_1, \Phi_2, \ldots, \Phi_{k-1}$, this minimum

will be the $k$-th eigenvalue $\lambda_k$ and the function on which it is reached will be the corresponding eigenfunction $\Phi_k$. The proof of this is similar to that given above.

*Property 7* The eigenvalues do not increase as $k(M)[q(M)]$ increases. More precisely, if $k_1(M) \geqslant k_2(M)$ in $\bar{D}$, then $\lambda_n^{(1)} \geqslant \lambda_n^{(2)}$. We shall give the proof for $\lambda_1$.

For any function $\Phi \in A$ we have

$$\frac{R_1[\Phi, \Phi]}{\|\Phi\|^2} \geqslant \frac{R_2[\Phi, \Phi]}{\|\Phi\|^2}$$

where $R_1$ and $R_2$ are the functionals $R$ corresponding to the functions $k_1(M)$ and $k_2(M)$. Consequently,

$$\lambda_1^{(1)} = \inf_{\Phi \in A} \frac{R_1[\Phi, \Phi]}{\|\Phi\|^2} \geqslant \inf_{\Phi \in A} \frac{R_2[\Phi, \Phi]}{\|\Phi\|^2} = \lambda_1^{(2)}$$

For the case $q_1(M) \geqslant q_2(M)$ the proof is practically the same as before.

*Property 8* The eigenvalues do not increase with increasing $\rho(M)$. More precisely, if $\rho_1(M) \geqslant \rho_2(M)$ in $\bar{D}$, then $\lambda_n^{(1)} \leqslant \lambda_n^{(2)}$.

We shall give the proof for $\lambda_1$. For any function $\Phi$ belonging to class $A$ we have

$$\frac{R[\Phi, \Phi]}{\|\Phi\|_{\rho_1}^2} \leqslant \frac{R[\Phi, \Phi]}{\|\Phi\|_{\rho_2}^2}$$

where $\|\Phi\|_{\rho_1}$ and $\|\Phi\|_{\rho_2}$ are the norms of the function $\Phi$ with weights $\rho_1$ and $\rho_2$. We then have

$$\lambda_1^{(1)} = \inf_{\Phi \in A} \frac{R[\Phi, \Phi]}{\|\Phi\|_{\rho_1}^2} \leqslant \inf_{\Phi \in A} \frac{R[\Phi, \Phi]}{\|\Phi\|_{\rho_2}^2} = \lambda_1^{(2)}$$

which was to be proved.

From Properties 7 and 8 it follows that in the one-dimensional case the eigenvalues $\lambda_n$ increase as $n^2$ with increasing $n$. In point of fact, in addition to the equation

$$\frac{\mathrm{d}}{\mathrm{d}x}[k(x)\Phi'(x)] - q(x)\Phi(x) + \lambda\rho(x)\Phi(x) = 0 \tag{27}$$

let us consider the equations

$$k_2\Phi'' + (\lambda\rho_1 - q_2)\Phi = 0 \tag{28}$$

and

$$k_1\Phi'' + (\lambda\rho_2 - q_1)\Phi = 0 \tag{29}$$

where $k_2, q_2, \rho_2$ are the maximum values of the functions $k(x)$, $q(x)$, $\rho(x)$ in the range $[0, l]$; $k_1, q_1$ and $\rho_1$ are their minimum values (either sup or inf).

To be specific, consider the first boundary-value problem, i.e. let us determine the solutions of equations (27), (28) and (29) subject to the boundary conditions

$$\Phi(0) = \Phi(l) = 0 \tag{30}$$

Since (28) and (29) have constant coefficients, the eigenvalues of the problems (28)–(30) and (29)–(30) can readily be found. They are given by

$$\lambda_n'' = \frac{\pi^2 n^2}{\rho_1 l^2} k_2 + \frac{q_2}{\rho_1}, \qquad \lambda_n' = \frac{\pi^2 n^2}{\rho_2 l^2} k_1 + \frac{q_1}{\rho_2}$$

From Properties 7 and 8, the eigenvalues $\lambda_n$ for the problem (27)–(30) lie between $\lambda_n'$ and $\lambda_n''$, i.e.

$$\lambda_n' \leqslant \lambda_n \leqslant \lambda_n''$$

This confirms the validity of the above statement. The fact that there are an infinite number of eigenvalues and eigenfunctions in the one-dimensional case (Theorem 1, Section 4.1.1) is also a consequence of these inequalities. The reader is recommended to give the proof for the second and third boundary-value problems.

*Property 9* The eigenvalues do not decrease with decreasing domain $D$, i.e. if $D' \subset D''$, then $\lambda_n' \geqslant \lambda_n''$.

We shall give the proof of this property only for $\lambda_1$ for the first boundary-value problem. To each domain $D'$ and $D''$ there correspond their own classes of functions $A'$, $A''$.

Suppose that some function $\Phi'$ belongs to class $A'$. It is zero (in view of the boundary conditions on the boundary of $D'$) on that part $\Sigma'$ of the boundary of $D'$ which is contained in $D''$ (Fig. 4.1).

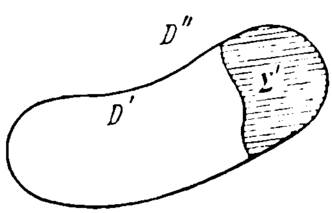



*Fig. 4.1*

The function $\Phi''$ which is equal to $\Phi'$ in $\bar{D}'$ and zero in $\bar{D}''-\bar{D}'$ (shaded region) will clearly belong to class $A''$. If we perform this operation for each of the functions in $A'$, we obtain the new class

of functions $\tilde{A}'$ which is a part of $A''$. For any function $\Phi \in A'$ we have

$$R''[\Phi, \Phi] = -\int_{D''} \Phi L[\Phi]\,\mathrm{d}\tau = -\int_{D'} \Phi L[\Phi]\,\mathrm{d}\tau = R'[\Phi, \Phi]$$

and

$$\int_{D''} \rho \Phi^2\,\mathrm{d}\tau = \int_{D'} \rho \Phi^2\,\mathrm{d}\tau$$

since this function is identically equal to zero in $\bar{D}''-\bar{D}'$. Therefore,

$$\lambda_1' = \inf_{\Phi \in A'} \frac{R'[\Phi, \Phi]}{\|\Phi\|_1^2} = \inf_{\Phi \in \tilde{A}'} \frac{R''[\Phi, \Phi]}{\|\Phi\|_2^2} \geqslant \inf_{\Phi \in A''} \frac{R''[\Phi, \Phi]}{\|\Phi\|_2^2} = \lambda_1''$$

where $\|\Phi\|_1$ and $\|\Phi\|_2$ are the norms of $\Phi$ in $D'$ and $D''$, respectively.

*Definition* An eigenvalue $\lambda$ will be called $r$-fold degenerate if there exist $r$ linearly independent eigenfunctions with this eigenvalue.

*Definition* An eigenvalue $\lambda$ will be called a simple (or non-degenerate) eigenvalue if any two eigenfunctions with this particular $\lambda$ are linearly dependent.

*Property 10* All eigenvalues of the one-dimensional boundary-value problem (5)–(6) are simple eigenvalues.

*Proof* Suppose that $\Phi_1(x)$ and $\Phi_2(x)$ are eigenfunctions corresponding to the same real eigenvalue $\lambda$. Both these functions are then the solutions of the same equation

$$\frac{\mathrm{d}}{\mathrm{d}x}[k\Phi']-q\Phi+\lambda\rho\Phi = 0$$

and satisfy the same boundary conditions on the left-hand side

$$\gamma_1\Phi_1'(0)-\gamma_2\Phi_1(0) = 0$$

$$\gamma_1\Phi_2'(0)-\gamma_2\Phi_2(0) = 0$$

These can be regarded as a system of linear equations for $\gamma_1$ and $\gamma_2$. Since $\gamma_1^2+\gamma_2^2 \neq 0$, the determinant of the system is zero. However, this is the Wronskian $W(x)$ for the solutions $\Phi_1(x)$ and $\Phi_2(x)$ at $x = 0$. It is well known that the Wronskian consisting of the solutions of a given linear homogeneous equation is either identically zero or does not vanish anywhere. Since, in our case, $W(0) = 0$, it follows that $W(x) \equiv 0$. This means that $\Phi_1(x)$ and $\Phi_2(x)$ must

be linearly dependent. We note that in multi-dimensional boundary-value problems this is no longer true.

*Example 5* Consider the vibrations of a square membrane with fixed edges under the action of an initial excitation. The edges of the square lie along the coordinate axes.

Mathematically this problem may be formulated as follows:

$$a^2 \nabla^2 u = u_{tt}, \qquad u = u(x, y, t) \tag{31}$$

$$u(0, y, t) = u(l, y, t) = 0 \tag{32}$$

$$u(x, 0, t) = u(x, l, t) = 0 \tag{33}$$

$$u(x, y, 0) = \varphi(x, y), \qquad u_t(x, y, 0) = \varphi_1(x, y) \tag{34}$$

Among functions of the form $\Phi(x, y)\, \Psi(t)$ we shall seek solutions of (31) which satisfy only the boundary conditions (32), (33). Substituting this function into (31), (32) and (33) and separating the variables, we obtain the following Sturm–Liouville problem

$$\nabla^2 \Phi + \lambda \Phi = 0 \tag{35}$$

$$\Phi(0, y) = \Phi(l, y) = 0 \tag{36}$$

$$\Phi(x, 0) = \Phi(x, l) = 0 \tag{37}$$

This problem can also be solved by the method of separation of variables. We shall seek solutions of the form $\Phi(x, y) = A(x)\, B(y)$. Substituting this function into (35) and separating the variables we obtain

$$\frac{A''}{A} + \frac{B''}{B} + \lambda = 0$$

In order that this should be an identity it is necessary that $A''/A = -\mu$ and $B''/B + \lambda = \mu$, i.e.

$$A'' + \mu A = 0 \tag{38}$$

$$B'' + (\lambda - \mu) B = 0 \quad \text{where} \quad B'' + \alpha B = 0 \tag{39}$$

From (36) and (37) we find that

$$A(0) = A(l) = 0 \tag{40}$$

$$B(0) = B(l) = 0 \tag{41}$$

Therefore, we have the boundary-value problems (38)–(40) and (39)–(41). The eigenvalues $\mu$ and $\lambda-\mu$ should be positive (Property 5). As in Example 1, we find that

$$\mu_n = \frac{\pi^2 n^2}{l^2} \quad (n = 1, 2, \ldots)$$

$$A_n(x) = \sin\frac{\pi n}{l}x$$

and

$$\alpha_k = \frac{\pi^2 k^2}{l^2} \quad (k = 1, 2, \ldots)$$

$$B_k(y) = \sin\frac{\pi k}{l}y$$

However, $\alpha_k = \lambda - \mu_n$ and, consequently,

$$\lambda_{n,k} = \alpha_k + \mu_n \qquad \text{or} \qquad \lambda_{n,k} = \frac{\pi^2}{l^2}(n^2+k^2)$$

where $k$ and $n$ assume the values $1, 2, \ldots$ independently of each other. We have thus found the eigenvalues of the problem (35)–(37). The corresponding eigenfunctions are

$$\Phi_{n,k}(x, y) = \sin\frac{\pi n}{l}x \sin\frac{\pi k}{l}y$$

The eigenvalues $\lambda_{n,k}$ and $\lambda_{k,n}$ will clearly be equal and the corresponding eigenfunctions

$$\Phi_{n,k} = \sin\frac{\pi n}{l}x \sin\frac{\pi k}{l}y \quad \text{and} \quad \Phi_{k,n} = \sin\frac{\pi k}{l}x \sin\frac{\pi n}{l}y$$

are linearly independent. For example, $\lambda_{1,2} = \lambda_{2,1} = 5(\pi^2/l^2)$

$$\Phi_{1,2} = \sin\frac{\pi}{l}x \sin\frac{2\pi}{l}y \quad \text{and} \quad \Phi_{2,1} = \sin\frac{2\pi}{l}x \sin\frac{\pi}{l}y$$

Therefore, the eigenvalues of this particular problem are not simple eigenvalues.

The solution of (31)–(34) can be represented by the series

$$u(x, y, t) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(C_{n,k}\cos a\sqrt{\lambda_{n,k}}\,t$$

$$+ D_{n,k}\sin a\sqrt{\lambda_{n,k}}\,t)\sin\frac{\pi n}{l}x \sin\frac{\pi k}{l}y$$

in which the coefficients are given by

$$C_{n,k} = \frac{4}{l^2}\int_0^l\int_0^l \varphi(\xi, \eta) \sin\frac{\pi n}{l}\xi \sin\frac{\pi k}{l}\eta \, d\xi \, d\eta$$

$$D_{n,k} = \frac{4}{al^2\sqrt{\lambda_{n,k}}}\int_0^l\int_0^l \varphi_1(\xi, \eta) \sin\frac{\pi n}{l}\xi \sin\frac{\pi k}{l}\eta \, d\xi \, d\eta$$

## 4.2 SOME PROPERTIES OF A SET OF EIGENFUNCTIONS

We shall now consider some of the properties of a set of eigenfunctions $\{\Phi_n\}$.

*Definition* A set of functions $\{\Phi_n\}$ which are orthogonal in pairs in a given domain (with a weight $\rho$) is called complete in $\bar{D}$ if for any function $f(M)$, which is square integrable in $\bar{D}$,

$$\int_D \rho f^2 \, d\tau = \sum_{k=1}^{\infty} C_k^2 \|\Phi_k\|^2 \tag{42}$$

where $C_k$ are the Fourier coefficients of the expansion of $f(M)$ in terms of the functions $\{\Phi_k\}$.

*A sufficient test for the completeness of the system* $\{\Phi_n\}$ If a function $F(M)$ is continuous in $\bar{D}$, and for any $\varepsilon > 0$ there exists a linear combination $S_n = \alpha_1\Phi_1 + \ldots + \alpha_n\Phi_n$ for which $\int_D \rho(F-S_n)^2 \, d\tau < \varepsilon$, then the set $\{\Phi_n\}$ is complete.

Consider a fixed $\varepsilon > 0$. For any function $f(M)$ which is square integrable in $\bar{D}$, we can find a function $\varphi(M)$ which is continuous in $\bar{D}$ and is such that $\int_D \rho(f-\varphi)^2 \, d\tau < \varepsilon/4$. This requires proof but we shall not give it here. For the function $\varphi(M)$ and for the chosen $\varepsilon$, it is possible by hypothesis to find a linear combination $S_n = \alpha_1\Phi_1 + \ldots + \alpha_n\Phi_n$, for which

$$\int_D \rho(\varphi - S_n)^2 \, d\tau < \frac{\varepsilon}{4}$$

Consider the integral

$$\int_D \rho\Big(f - \sum_{k=1}^{n} C_k\Phi_k\Big)^2 \, d\tau$$

in which $\sum_{k=1}^{n} C_k\Phi_k$ is the partial sum of the Fourier series for $f(M)$. It is clear that

$$\int_D \rho\left(f-\sum_{k=1}^{n} C_k\Phi_k\right)^2 \mathrm{d}\tau = \int_D \rho f^2\,\mathrm{d}\tau - 2\sum_{k=1}^{n} C_k \int_D \rho f\Phi_k\,\mathrm{d}\tau + \sum_{k=1}^{n} C_k^2\|\Phi_k\|^2$$

where we use the orthogonality property of the function $\Phi_k$. Since $\int_D \rho f\Phi_k\,\mathrm{d}\tau = C_k\|\Phi_k\|^2$, it follows that

$$\int_D \rho\left(f-\sum_{k=1}^{n} C_k\Phi_k\right)^2 \mathrm{d}\tau = \int_D \rho f^2\,\mathrm{d}\tau - \sum_{k=1}^{n} C_k^2\|\Phi_k\|^2$$

It is well known that the quantity $\delta_n^2 = \int_D \rho(f-S_n)^2\,\mathrm{d}\tau$ is a minimum if $S_n$ is taken to be $\sum_{k=1}^{n} C_k\Phi_k$. Therefore

$$\begin{aligned}0 &\leqslant \int_D \rho f^2\,\mathrm{d}\tau - \sum_{k=1}^{n} C_k^2\|\Phi_k\|^2 = \int_D \rho\left(f-\sum_{k=1}^{n} C_k\Phi_k\right)^2 \mathrm{d}\tau\\ &\leqslant \int_D \rho(f-S_n)^2\,\mathrm{d}\tau \leqslant \int_D \rho(f-\varphi+\varphi-S_n)^2\,\mathrm{d}\tau\\ &\leqslant 2\int_D \rho(f-\varphi)^2\,\mathrm{d}\tau + 2\int_D \rho(\varphi-S_n)^2\,\mathrm{d}\tau < 2\frac{\varepsilon}{4} + 2\frac{\varepsilon}{4} = \varepsilon\end{aligned}$$

In this expression we have used the well-known inequality

$$(A+B)^2 \leqslant 2A^2+2B^2$$

It follows that

$$0 \leqslant \int_D \rho f^2\,\mathrm{d}\tau - \sum_{k=1}^{n} C_k^2\|\Phi_k\|^2 < \varepsilon$$

and hence we have the completeness condition

$$\int_D \rho f^2\,\mathrm{d}\tau = \sum_{k=1}^{\infty} C_k^2\|\Phi_k\|^2$$

Expansions of the Fourier type in terms of the complete systems of functions $\{\Phi_n\}$ have the following remarkable property.

*Theorem* If a set of functions $\{\Phi_n\}$ is complete in a domain $\bar{D}$, then the series of the Fourier type for any function which is square integrable in $D$ can be integrated term by term independently of whether it is convergent or not, i.e. for any domain $D' \subset D$

$$\int_{D'} f(M)\,\mathrm{d}\tau = \sum_{n=1}^{\infty} C_n \int_{D'} \Phi_n(M)\,\mathrm{d}\tau$$

*Proof* Consider the difference $\delta_n = \int_{D'} f\,\mathrm{d}\tau - \sum_{k=1}^{n} C_k \int_{D'} \Phi_k\,\mathrm{d}\tau$

$$|\delta_n| = \left|\int_{D'}\left(f - \sum_{k=1}^{n} C_k\Phi_k\right)\mathrm{d}\tau\right|$$

$$\leqslant \int_{D'}\left|f - \sum_{k=1}^{n} C_k\Phi_k\right|\mathrm{d}\tau \leqslant \int_{D}\left|f - \sum_{k=1}^{n} C_k\Phi_k\right|\mathrm{d}\tau$$

To estimate the last integral we shall use the Cauchy inequality

$$\int_D \left|f - \sum_{k=1}^{n} C_k\Phi_k\right|\mathrm{d}\tau = \int_D \sqrt{\rho}\left|f - \sum_{k=1}^{n} C_k\Phi_k\right|\frac{\mathrm{d}\tau}{\sqrt{\rho}}$$

$$\leqslant \sqrt{\int_D \rho\left(f - \sum_{k=1}^{n} C_k\Phi_k\right)^2\mathrm{d}\tau}\,\sqrt{\int_D \frac{\mathrm{d}\tau}{\rho}}$$

$$= \sqrt{\int_D \rho f^2\,\mathrm{d}\tau - \sum_{k=1}^{n} C_k^2\|\Phi_k\|^2}\,\sqrt{\int_D \frac{\mathrm{d}\tau}{\rho}}$$

The last integral is bounded and the difference

$$\int_D \rho f^2\,\mathrm{d}\tau - \sum_{k=1}^{n} C_k^2\|\Phi_k\|^2$$

tends to zero as $n \to \infty$ in view of the completeness condition. Consequently, $\delta_n \to 0$ as $n \to \infty$, which was to be proved.

*Theorem* The set of eigenfunctions of the boundary value problem (5)–(6) is complete.

*Proof* Consider the function $f(M)$ which is arbitrary and continuous in $\bar{D}$. For any number $\varepsilon > 0$, the functions belonging to class $A$ (Section 4.1) will include a function $g(M)$ such that

$$\int_D \rho (f-g)^2 \, d\tau < \frac{\varepsilon}{4} \tag{43}$$

The series of the Fourier type $g(M) = \sum_{k=1}^{\infty} C_k \Phi_k$ converges uniformly in $\bar{D}$. Consequently, for any $\varepsilon_1 > 0$ we can find $N(\varepsilon_1)$ such that

$$\left| g - \sum_{k=1}^{n} C_k \Phi_k \right| < \varepsilon_1 \quad \text{for} \quad n > N(\varepsilon_1) \tag{44}$$

We shall show $\int_D \rho \left( f - \sum_{k=1}^{n} C_k \Phi_k \right)^2 d\tau < \varepsilon$. By the above test for completeness $\{\Phi_n\}$ must be complete. It is evident that

$$\int_D \rho \left( f - \sum_{k=1}^{n} C_k \Phi_k \right)^2 d\tau = \int_D \rho \left( f - g + g - \sum_{k=1}^{n} C_k \Phi_k \right)^2 d\tau$$

$$\leqslant 2 \int_D \rho (f-g)^2 \, d\tau + 2 \int_D \rho \left( g - \sum_{k=1}^{n} C_k \Phi_k \right)^2 d\tau$$

Using (43) and (44) we obtain

$$\int_D \rho \left( f - \sum_{k=1}^{n} C_k \Phi_k \right)^2 d\tau < \frac{\varepsilon}{2} + 2\varepsilon_1^2 \int_D \rho \, d\tau < \varepsilon$$

where $\varepsilon_1 \leqslant \sqrt{\varepsilon}/2\sqrt{B}$ and $B = \int_D \rho \, d\tau$.

The last two theorems enable us to integrate the series expansion in terms of the eigenfunctions of the boundary-value problem (5)–(6) term by term for any function which is square integrable without bothering either about the uniform convergence of these series, or even whether they converge at all.

## 4.3 SOLUTION OF INHOMOGENEOUS BOUNDARY-VALUE PROBLEMS BY THE FOURIER METHOD

Knowledge of the set of eigenfunctions $\{\Phi_n\}$ and the corresponding eigenvalues $\{\lambda_n\}$ enables us to solve inhomogeneous boundary-value problems. Let us consider some of these.

**4.3.1** It is required to find the solution of the problem

$$L[u]+f(M, t) = \rho u_{tt} \quad \text{(respectively } u_t\text{)} \tag{45}$$

$$u(M, 0) = 0, \qquad u_t(M, 0) = 0 \tag{46}$$

$$\left(\gamma_1 \frac{\partial u}{\partial n} + \gamma_2 u\right)_S = 0 \tag{47}$$

which is continuous in the closed domain $\bar{B} \equiv \{M \in \bar{D}, t \geqslant 0\}$.

The required solution $u(M, t)$ belongs to class $A$ and, therefore, in accordance with the expansion theorem it can be represented by series of the Fourier type in terms of the eigenfunctions $\{\Phi_n\}$ corresponding to the inhomogeneous problem (5)–(6):

$$u(M, t) = \sum_{n=1}^{\infty} \Psi_n(t)\,\Phi_n(M) \tag{48}$$

where

$$\Psi_n(t) = \frac{1}{\|\Phi_n\|^2} \int_D \rho u(P, t)\,\Phi_n(P)\,\mathrm{d}\tau \tag{49}$$

Using Equation (5) for $\rho\Phi_n$ under the integral sign in (49), we obtain

$$\Psi_n(t) = -\frac{1}{\lambda_n \|\Phi_n\|^2} \int_D uL[\Phi_n]\,\mathrm{d}\tau = \frac{R[u, \Phi_n]}{\lambda_n \|\Phi_n\|^2}$$

$$= \frac{R[\Phi_n, u]}{\lambda_n \|\Phi_n\|^2} = -\frac{1}{\lambda_n \|\Phi_n\|^2} \int_D \Phi_n L[u]\,\mathrm{d}\tau$$

Substituting for $L[u]$ from (45), we obtain

$$\Psi_n(t) = -\frac{1}{\lambda_n \|\Phi_n\|^2} \int_D \rho u_{tt} \Phi_n\,\mathrm{d}\tau + \frac{1}{\lambda_n \|\Phi_n\|^2} \int_D f\Phi_n\,\mathrm{d}\tau \tag{50}$$

The first term on the right-hand side of (50) is equal to $-\Psi_n''/\lambda_n$. The second term is a known function and will be represented by $f_n(t)1/\lambda_n$. Therefore,

$$\Psi_n(t) \equiv -\frac{\Psi_n''}{\lambda_n} + \frac{f_n}{\lambda_n}$$

Consequently, $\Psi_n(t)$ is the solution of the equation

$$\Psi_n'' + \lambda_n \Psi_n = f_n(t) \quad \text{(respectively } \Psi_n' + \lambda_n \Psi_n = f_n\text{)}$$

with the additional conditions

$$\Psi_n(0) = \frac{1}{\|\Phi_n\|^2} \int_D \rho\, u(P, 0) \Phi_n(P)\, d\tau = 0$$

$$\Psi_n'(0) = \frac{1}{\|\Phi_n\|^2} \int_D \rho\, u_t(P, 0) \Phi_n(P)\, d\tau = 0$$

The solution of this problem is of the form

$$\Psi_n(t) = \frac{1}{\sqrt{\lambda_n}} \int_0^t \sin \sqrt{\lambda_n}\, (t-\theta) f_n(\theta)\, d\theta$$

$$\left(\text{respectively } \Psi_n(t) = \int_0^t e^{-\lambda_n(t-\theta)} f_n(\theta)\, d\theta\right)$$

Substituting the functions $\Psi_n(t)$ obtained above into (48), we obtain the required solution in the form of the series expansion in the eigenfunctions. If, in particular, $f(M, t) = f(t)\, \delta(M, M_0)$, then

$$f_n(t) = \frac{1}{\|\Phi_n\|^2} \int_D f(t)\, \delta(P, M_0) \Phi_n(P)\, d\tau = \frac{\Phi_n(M_0)}{\|\Phi_n\|^2} f(t)$$

and

$$\Psi_n(t) = \frac{\Phi_n(M_0)}{\sqrt{\lambda_n}\, \|\Phi_n\|^2} \int_0^t \sin \sqrt{\lambda_n}(t-\theta) f(\theta)\, d\theta$$

$$\left(\text{respectively } \Psi_n(t) = \frac{\Phi_n(M_0)}{\|\Phi_n\|^2} \int_0^t e^{-\lambda_n(t-\theta)} f(\theta)\, d\theta\right)$$

Consider the problem

$$L[u] + f(M, t) = \rho u_{tt} \qquad (\rho u_t)$$

$$u(M, 0) = \varphi(M), \qquad u_t(M, 0) = \varphi_1(M)$$

$$\left(\gamma_1 \frac{\partial u}{\partial n} + \gamma_2 u\right)_S = 0$$

We shall seek the solution in the form of the sum

$$u(M, t) = v(M, t) + w(M, t)$$

where $v(M, t)$ and $w(M, t)$ are the solutions of the following problems:

$$v: \qquad L[v] = \rho v_{tt} \qquad (\rho v_t)$$

$$v(M, 0) = \varphi(M), \qquad v_t(M) = \varphi_1(M), \qquad \left(\gamma_1 \frac{\partial v}{\partial n} + \gamma_2 v\right)_S = 0$$

$$w: \qquad L[w] + f(M, t) = \rho w_{tt} \qquad (\rho w_t)$$

$$w(M, 0) = w_t(M, 0) = 0, \qquad \left(\gamma_1 \frac{\partial w}{\partial n} + \gamma_2 w\right)_S = 0$$

We already know how to solve these problems.

**4.3.2** Suppose it is required to find the solution of the problem

$$L[u] + f(M, t) = \rho u_{tt} \qquad (\rho u_t) \tag{51}$$

$$\left(\gamma_1 \frac{\partial u}{\partial n} + \gamma_2 u\right)_S = \beta(M, t) \tag{52}$$

$$u(M, 0) = \varphi(M), \qquad u_t(M, 0) = \varphi_1(M) \tag{53}$$

which is continuous in the closed domain $\bar{B} \equiv \{M \in \bar{D};\ t \geqslant 0\}$. We shall consider the following method of solving this problem. Among the functions $v(M, t)$ which are continuous in $\bar{B}$ and have in this domain piecewise continuous first- and second-order partial derivatives, we shall take the function $v_1(M, t)$ which satisfies the prescribed boundary conditions given by (52). We shall seek the solution $u(M, t)$ in the form of the sum $u = v_1(M, t) + w(M, t)$, where $w(M, t)$ is continuous in $\bar{B}$. The problem can then be formulated as follows:

$$L[w] + f_1(M, t) = \rho w_{tt} \qquad (\rho w_t)$$

$$w(M, 0) = \bar{\varphi}(M), \qquad w_t(M, 0) = \bar{\varphi}_1(M)$$

$$\left(\gamma_1 \frac{\partial w}{\partial n} + \gamma_2 w\right)_S = 0$$

where

$$f_1(M, t) = f(M, t) + L[v_1] - \rho v_{1tt}$$

$$\bar{\varphi}(M) = \varphi(M) - v_1(M, 0), \qquad \bar{\varphi}_1(M) = \varphi_1(M) - v_{1t}(M, 0)$$

We have already considered this problem in Section 4.3.1. The form of the function $v_1(M, t)$ is either guessed or it is found by Duhamel's method (see Section 4.3.3).

*Example 6* It is required to solve the problem

$$a^2u_{xx} = u_{tt}$$

$$u(x, 0) = \varphi_1(x), \qquad u_t(x, 0) = \varphi_2(x)$$

$$u(0, t) = \mu_1(t), \qquad u(l, t) = \mu_2(t)$$

We shall assume that the functions $\mu_1(t)$ and $\mu_2(t)$ are twice differentiable and will take

$$v_1(x, t) = \frac{l-x}{l}\mu_1(t)+\frac{x}{l}\mu_2(t)$$

The solution will be sought in the form of the sum

$$u(x, t) = v_1(x, t)+w(x, t)$$

The function $w(x, t)$ will clearly be the solution of the following problem:

$$a^2w_{xx}+\frac{x-l}{l}\mu_1''(t)-\frac{x}{l}\mu_2''(t) = w_{tt}$$

$$w(x, 0) = \varphi_1(x)+\frac{x-l}{l}\mu_1(0)-\frac{x}{l}\mu_2(0) = \tilde{\varphi}_1(x)$$

$$w_t(x, 0) = \varphi_2(x)+\frac{x-l}{l}\mu_1'(0)-\frac{x}{l}\mu_2'(0) = \tilde{\varphi}_2(x)$$

$$w(0, t) = w(l, t) = 0$$

The function $w(x, t)$ will also be sought in the form of the sum $w = R(x, t)+Q(x, t)$, where $R(x, t)$ is the solution of the homogeneous boundary-value problem

$$a^2R_{xx} = R_{tt}$$

$$R(x, 0) = \tilde{\varphi}_1(x), \qquad R_t(x, 0) = \tilde{\varphi}_2(x), \qquad R(0, t) = R(l, t) = 0$$

and is given by (see Example 1)

$$R(x, t) = \sum_{n=1}^{\infty}(C_n\cos a\sqrt{\lambda_n}t+D_n\sin a\sqrt{\lambda_n}t)\sin\frac{\pi n}{l}x$$

$$\lambda_n = \frac{\pi^2n^2}{l^2}, \qquad C_n = \frac{2}{l}\int_0^l \tilde{\varphi}_1(\xi)\sin\frac{\pi n}{l}\xi\, d\xi$$

$$D_n = \frac{2}{a\pi n}\int_0^l \tilde{\varphi}_2(\xi)\sin\frac{\pi n}{l}\xi\, d\xi$$

and $Q(x, t)$ is the solution of the following problem

$$a^2Q_{xx}+f(x, t) = Q_t.$$

$$Q(x, 0) = Q_t(x, 0) = 0$$

$$Q(0, t) = Q(l, t) \quad = 0$$

where $f(x, t) = \frac{x-l}{l}\mu_1''(t) - \frac{x}{l}\mu_2''(t)$.

According to Section 4.3.1,

$$Q(x, t) = \sum_{n=1}^{\infty} \Psi_n(t) \sin\frac{\pi n}{l}x$$

The functions $\Psi_n(t)$ are given by

$$\Psi_n(t) = \frac{1}{\sqrt{\lambda_n}} \int_0^t \sin\sqrt{\lambda_n}(t-\theta) f_n(\theta)\,\mathrm{d}\theta$$

where

$$f_n(\theta) = \frac{2}{l}\int_0^l f(\xi, t)\sin\frac{\pi n}{l}\xi\,\mathrm{d}\xi = \frac{2}{\pi n}[(-1)^n\mu_2''(\theta) - \mu_1''(\theta)]$$

*Remark 1* If the boundary conditions are of the form

$$\alpha_1 u_x(0, t) - \beta_1 u(0, t) = \mu_1(t), \qquad \alpha_2 u_x(l, t) + \beta_2 u(l, t) = \mu_2(t)$$

the function $v_1(x, t)$, which satisfies these boundary conditions, can be taken to be

$$v(x, t) = Dx^2\mu_2(t) - C(x-l)^2\mu_1(t)$$

where

$$C = \frac{1}{2\alpha_1 l + \beta_1 l^2}, \qquad D = \frac{1}{2\alpha_2 l + \beta_2 l^2}$$

*Remark 2* It is sometimes easy to find the function $v_1(x, t)$ which satisfies not only the prescribed inhomogeneous boundary conditions but also a given equation.

*Example 7* It is required to solve the problem

$$a^2u_{xx} = u_{tt} \tag{54}$$

$$u(x, 0) = \varphi_1(x), \quad u_t(x, 0) = \varphi_2(x) \tag{55}$$

$$u(0, t) = 0, \qquad u(l, t) = A\sin\omega t \quad \left(\frac{\omega}{a}l \neq n\pi\right) \tag{56}$$

Among functions of the form $F(x)\sin\omega t$ it is quite easy to find the solution $v_1(x, t)$ of (54), which satisfies the boundary conditions (56). In fact, substituting this function into (54) and dividing both sides of the equation by $\sin\omega t$, we obtain the equation for $F(x)$:

$$a^2F''+\omega^2F = 0 \tag{57}$$

From the boundary conditions (56) we find that

$$F(0) = 0, \qquad F(l) = A \tag{58}$$

The solution of the problem defined by (57)–(58) will be of the form

$$F(x) = A\frac{\sin\frac{\omega}{a}x}{\sin\frac{\omega}{a}l}$$

and, consequently,

$$v_1(x, t) = A\frac{\sin\frac{\omega}{a}x}{\sin\frac{\omega}{a}l}\sin\omega t$$

The solution of the problem (54)–(56) will be sought in the form

$$u(x, t) = v_1(x, t)+w(x, t)$$

where $w(x, t)$ is the solution of the following problem:

$$a^2w_{xx} = w_{tt}$$

$$w(0, t) = 0, \qquad w(l, t) = 0$$

$$w(x, 0) = \varphi_1(x), \qquad w_t(x, 0) = \varphi_2(x)-A\omega\frac{\sin\frac{\omega}{a}x}{\sin\frac{\omega}{a}l} = \bar{\varphi}_2(x)$$

This inhomogeneous boundary-value problem can be solved by the method of separation of variables.

**4.3.3** *Duhamel's method* This can be used for boundary-value problems with inhomogeneous boundary conditions which can be reduced to the form

$$L[u] = \rho(M)\begin{cases} u_{tt} \\ u_t \end{cases} \tag{59}$$

$$u(M,0) = u_t(M,0) = 0 \quad [\text{respectively} \quad u(M,0) = 0] \tag{60}$$

$$\left(\alpha\frac{\partial u}{\partial n}+\beta u\right)_S = \mu(M,t) \tag{61}$$

This involves the following steps.

1. First we solve the problem defined by (59)–(61) with a stationary inhomogeneity on the boundary condition, i.e. with a boundary condition of the form

$$\left(\alpha\frac{\partial u}{\partial n}+\beta u\right)_S = \eta(t)\mu(M,\tau)$$

[the stationary inhomogeneity $\mu(M,\tau)$ on the boundary is switched on at time $t = 0$], where $\tau$ is a fixed number. Suppose that $w(M, t, \tau)$ is the solution of this problem, which is continuous together with the first-order derivatives and the derivative $w_{tt}$ in the domain $(M \in D,\ t \geqslant 0)$. The solution of this problem must be treated as a generalised function because $\eta(t)\,\mu(M,\tau)$ is a generalised function.

The solution of the problem (59)–(61), subject to the boundary and the initial conditions of the form

$$\left(\alpha\frac{\partial u}{\partial n}+\beta u\right)_S = \eta(t-\tau)\mu(M,\tau), \qquad u|_{t=\tau} = u_t|_{t=\tau} = 0$$

[the stationary inhomogeneity $\mu(M,\tau)$ is switched on at $t = \tau$] will be the function $w(M, t-\tau, \tau)\,\eta(t-\tau)$. We note that at internal points $M$ of $D$ we have

$$w(M,0,t) \equiv w_t(M,0,t) \equiv 0 \tag{62}$$

2. The solution of the problem defined by (59)–(61), subject to the boundary and initial conditions

$$u|_{t=\tau} = u_t|_{t=\tau} = 0$$

$$\left(\alpha\frac{\partial u}{\partial n}+\beta u\right)_S = \mu(M, \tau)[\eta(t-\tau)-\eta(t-\tau-d\tau)]$$

[stationary inhomogeneity $\mu(M, \tau)$ on the boundary exists only for the time interval between $t = \tau$ and $t = \tau+d\tau$] will be the function

$$w(M, t-\tau, \tau)\eta(t-\tau)-w(M, t-\tau, \tau)\eta(t-\tau-d\tau)$$
$$= \frac{\partial}{\partial t}[w(M, t-\tau, \tau)\eta(t-\tau)]\, d\tau$$

3. In the original boundary-value problem (59)–(61) the inhomogeneity on the boundary acts only for the time interval between 0 and $t$. One would therefore expect that the solution of (59)–(61) will be

$$u(M, t) = \int_0^t \frac{\partial}{\partial t}[w(M, t-\tau, \tau)\eta(t-\tau)]\, d\tau \tag{62a}$$

Direct verification will show that this is so. In point of fact, this function can also be written in the form

$$u(M, t) = \int_0^t \eta(t-\tau)w_t(M, t-\tau, \tau)\, d\tau+\int_0^t w(M, t-\tau, \tau)\,\delta(t-\tau)\, d\tau$$

since $(d/dt)\eta(t-\tau) = \delta(t-\tau)$. Since $\eta(t-\tau) = 1$ for all $\tau$ between 0 and $t$, we have, using the properties of the $\delta$-function,

$$u(M, t) = \int_0^t w_t(M, t-\tau, \tau)\, d\tau+w(M, 0, t) \tag{63}$$

From this expression and from the formulae for the derivative

$$u_t(M, t) = \int_0^t w_{tt}(M, t-\tau, \tau)\, d\tau+w_t(M, 0, t) \tag{64}$$

it follows directly that the initial conditions (60) are satisfied $[(w(M, 0, t) \equiv w_t(M, 0, t) \equiv 0$ for internal points of $D]$. The boundary condition (61) is also satisfied because

$$\left(\alpha\frac{\partial u}{\partial n}+\beta u\right)_S = \int_0^t \frac{\partial}{\partial t}\left\{\left(\alpha\frac{\partial w}{\partial n}+\beta w\right)_S \eta(t-\tau)\right\} \mathrm{d}\tau$$

$$= \int_0^t \frac{\partial}{\partial t}\{\mu(M, \tau)\eta^2(t-\tau)\} \,\mathrm{d}\tau$$

$$= \int_0^t \frac{\partial}{\partial t}\{\mu(M, \tau)\eta(t-\tau)\} \,\mathrm{d}\tau$$

$$= \int_0^t \mu(M, \tau)\frac{\mathrm{d}}{\mathrm{d}t}\eta(t-\tau) \,\mathrm{d}\tau$$

$$= \int_0^t \mu(M, \tau)\delta(t-\tau) \,\mathrm{d}\tau = \mu(M, t)$$

Let us substitute (62a) into (59) and use Equations (63) and (64). For internal points in $D$ we have, in view of (62),

$$u(M, t) = \int_0^t w_t(M, t-\tau, \tau)\,\mathrm{d}\tau$$

$$u_t(M, t) = \int_0^t w_{tt}(M, t-\tau, \tau)\,\mathrm{d}\tau$$

Consequently,

$$u_{tt} = \int_0^t w_{ttt}(M, t-\tau, \tau)\,\mathrm{d}\tau + w_{tt}(M, 0, t)$$

From the identity

$$L[w(M, t-\tau, \tau)] \equiv \rho(M) w_{tt}(M, t-\tau, \tau)$$

and using the continuity of $w_{tt}$ in the region $(M \in D,\ t \geqslant 0)$, we find that (for $t-\tau \to 0$)

$$L[w(M, 0, t)] = \rho(M) w_{tt}(M, 0, t) \equiv 0$$

since $w(M, 0, t) \equiv 0$ for internal points of $D$ and, consequently, $L[w(M, 0, t)] \equiv 0$.

Therefore,

$$u_{tt}(M, t) = \int_0^t w_{ttt}(M, t-\tau, \tau)\,\mathrm{d}\tau$$

so that

$$L[u]-\rho u_{tt} \equiv \int_0^t \frac{\partial}{\partial t}\{L[w]-\rho w_{tt}\}\,\mathrm{d}\tau \equiv 0$$

since $L[w] \equiv \rho w_{tt}$. This means that (62a) is the solution of (59)–(61).

Consider the special case $\mu(M, t) = Q(t)$. Proceeding by analogy with the above discussion, the solution can be obtained as follows.

1. We first solve Equation (59) subject to the boundary condition

$$\left(\alpha\frac{\partial u}{\partial n}+\beta u\right)_S = \eta(t)$$

i.e. for $Q(t) \equiv 1$.

Let $R(M, t)$ be the solution of this problem. The function $Q(\tau)\,R(M, t)$ will then be the solution of the problem with the boundary condition of the form

$$\left(\alpha\frac{\partial u}{\partial n}+\beta u\right)_S = \eta(t)\,Q(\tau)$$

where $\tau$ is a fixed number.

2. The function $Q(\tau)\,R(M, t-\tau)\,\eta(t-\tau)$ will be the solution of Equation (59) subject to the boundary and initial conditions of the form

$$\left(\alpha\frac{\partial u}{\partial n}+\beta u\right)_S = Q(\tau)\,\eta(t-\tau)$$

$$u|_{t=\tau} = u_t|_{t=\tau} = 0$$

We note that in view of the initial conditions, for all internal points of the domain $D$ we have

$$R(M, 0) \equiv R_t(M, 0) \equiv 0$$

3. The function

$$Q(\tau)\,[R(M, t-\tau)\eta(t-\tau)-R(M, t-\tau-\mathrm{d}\tau)\eta(t-\tau-\mathrm{d}\tau)]$$
$$= Q(\tau)\frac{\partial}{\partial t}[R(M, t-\tau)\eta(t-\tau)]\,\mathrm{d}\tau$$

will be the solution of (59) subject to the boundary and initial conditions of the form

$$\left(\alpha\frac{\partial u}{\partial n}+\beta u\right)_S = Q(\tau)\,[\eta(t-\tau)-\eta(t-\tau-\mathrm{d}\tau)]$$

$$u|_{t=\tau} = u_t|_{t=\tau} = 0$$

4. The solution of the original boundary-value problem will be the function

$$u(M,t) = \int_0^t Q(\tau)\frac{\partial}{\partial t}R(M,t-\tau)\,\mathrm{d}\tau$$

This can be verified by direct substitution as in the previous case. It is thus sufficient, in this case, to find the solution $R(M,t)$ for the problem with the very simple (stationary) inhomogeneity on the boundary condition $Q(t) \equiv 1$.

*Example 8* Find the solution of the problem

$$a^2u_{xx} = u_t, \qquad u(x,0) = 0, \qquad u(0,t) = 0, \qquad u(l,t) = Q(t)$$

To begin with, we find the solution $R(x,t)$ of the problem for $Q(t) \equiv 1$. The function $R(x,t)$ will be sought in the form of the sum $R = v(x)+P(x,t)$, in which $v(x)$ describes the steady-state conditions and $P(x,t)$ the departure from the steady state. For the function $v(x)$ the problem can be formulated as follows:

$$v'' = 0, \qquad v(0) = 0, \qquad v(l) = 1$$

The solution will be the function $x/l$. For $P(x,t)$ the problem can be formulated as follows:

$$a^2P_{xx} = P_t, \qquad P(x,0) = -\frac{x}{l}, \qquad P(0,t) = P(l,t) = 0$$

Solving this problem by the method of the separation of variables (see Example 1), we find that

$$P(x,t) = \sum_{n=1}^{\infty} C_n\,\mathrm{e}^{-a^2\lambda_n t}\sin\frac{\pi n}{l}x, \qquad \lambda_n = \frac{\pi^2n^2}{l^2}$$

The coefficients $C_n$ can be determined from the initial condition:

$$-\frac{x}{l} = \sum_{n=1}^{\infty} C_n\sin\frac{\pi n}{l}x$$

and are given by

$$C_n = \frac{2}{\pi}\frac{(-1)^n}{n}$$

Therefore,

$$R(x, t) = \frac{x}{l} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-a^2\lambda_n t} \sin\frac{\pi n}{l}x$$

Consequently, the solution of the original problem will be

$$u(x, t) = \int_0^t Q(\tau)\frac{\partial}{\partial t}[R(x, t-\tau)\eta(t-\tau)]\,d\tau$$

or

$$u(x, t) = \int_0^t Q(\tau)\frac{\partial}{\partial t}R(x, t-\tau)\,d\tau + \int_0^t Q(\tau)\,R(x, t-\tau)\,\delta(t-\tau)\,d\tau$$

$$= \int_0^t Q(\tau)\frac{\partial}{\partial t}R(x, t-\tau)\,d\tau + Q(t)R(x, 0)$$

The function $R(x, 0)$ is zero at all internal points within the range $[0, l]$ and for $x = 0$, while for $x = l$ we have $R(x, 0) = 1$.

If it is required to solve the problem

$$a^2u_{xx} = u_t, \qquad u(x, 0) = 0, \qquad u(0, t) = Q_1(t), \qquad u(l, t) = Q_2(t)$$

the solution can be sought in the form of the sum $u = v + w$, where for $v$ and $w$ we have the following two problems:

$$v:\quad a^2v_{xx} = v_t, \quad v(x, 0) = 0, \quad v(0, t) = Q_1(t), \quad v(l, t) = 0$$

$$w:\quad a^2w_{xx} = w_t, \quad w(x, 0) = 0, \quad w(0, t) = 0, \quad w(l, t) = Q_2(t)$$

Each of these problems can be solved by Duhamel's method, as shown in the above example. This method can also be used for the solution of boundary-value problems on a semi-infinite straight line.

## 4.4 UNIQUENESS OF THE SOLUTIONS OF BOUNDARY-VALUE PROBLEMS

**4.4.1** The uniqueness problem can be resolved as follows. If there are two continuous solutions of the problem (1)–(3) in a closed domain $\bar{B} \equiv \{M \in \bar{D}, t \geqslant 0\}$, their difference

$$v(M, t) = u_1(M, t) - u_2(M, t)$$

is the solution of the homogeneous boundary-value problem

$$L[v] = \rho v_{tt} \qquad (\text{or } \rho v_t)$$

$$v(M, 0) = v_t(M, 0) = 0$$

$$\left(\gamma_1 \frac{\partial v}{\partial n} + \gamma_2 v\right)_S = 0$$

In the class of functions $A$ this problem has the unique solution $v \equiv 0$. Consequently, $u_1 = u_2$. The conclusion that the solution of the homogeneous boundary-value problem for a function belonging to $A$ was based on the validity of the expansion theorem. However, the requirement that the expansion theorem is valid is not necessary for the uniqueness of the solution of boundary-value problems.

In the examples given below we shall prove the uniqueness of the solutions of boundary-value problems without referring to the expansion theorem.

*Theorem* The solution of the boundary-value problem

$$\frac{\partial}{\partial x}[k u_x] + f(x, t) = \rho u_{tt}$$

$$u(x, 0) = \varphi(x), \qquad u_t(x, 0) = \varphi_1(x)$$

$$\alpha_1 u_x(0, t) - \beta_1 u(0, t) = r_1(t), \qquad \alpha_2 u_x(l, t) + \beta_2 u(l, t) = r_2(t)$$

where $\alpha_i, \beta_i \geqslant 0\,(i = 1, 2)$, $k(x) > 0$, $\rho(x) > 0$, which is continuous in a closed domain $\{0 \leqslant x \leqslant l;\ t \geqslant 0\}$, is unique.

*Proof* Let $u_1(x, t)$ and $u_2(x, t)$ be two solutions of this problem. The function $v = u_1 - u_2$ is the solution of the homogeneous problem

$$\frac{\partial}{\partial x}(k v_x) = \rho v_{tt} \tag{65}$$

$$v(x, 0) = 0, \qquad v_t(x, 0) = 0, \qquad 0 \leqslant x \leqslant l \tag{66}$$

$$\alpha_1 v_x(0, t) - \beta_1 v(0, t) = 0, \qquad \alpha_2 v_x(l, t) + \beta_2 v(l, t) = 0 \tag{67}$$

and is continuous in the domain $\{0 \leqslant x \leqslant l,\ t \geqslant 0\}$. We shall show that $v(x, t) \equiv 0$. Consider the auxiliary function

$$E(t) = \frac{1}{2}\int_0^l [k(x) v_x^2(x, t) + \rho(x) v_t^2(x, t)]\,\mathrm{d}x \tag{68}$$

It can readily be shown that $E(t)$ is equal to the energy of vibrations described by the equation $(\partial/\partial x)(kv_x) = \rho v_{tt}$ for points within $[0, l]$ and is therefore called the energy integral. We shall show that $E(t) \equiv 0$. Consider the integral

$$E'(t) = \int_0^l (kv_x v_{xt} + \rho v_t v_{tt})\,\mathrm{d}x$$

Integrating the first term by parts, we have

$$E'(t) = kv_x v_t \Big|_0^l - \int_0^l v_t \left[\frac{\partial}{\partial x}(kv_x) - \rho v_{tt}\right] \mathrm{d}x$$

Since the function $v(x, t)$ satisfies Equation (65), the integrand is zero and, therefore,

$$E'(t) = k v_x v_t \Big|_0^l = k(l) v_x(l, t) v_t(l, t) - k(0) v_x(0, t) v_t(0, t) \tag{69}$$

If we are dealing with the first boundary-value problem, then $v_t(0, t) = v_t(l, t) \equiv 0$ and, consequently, $E'(t) \equiv 0$. If, on the other hand, we are dealing with the second boundary-value problem, then $v_x(l, t) = v_x(0, t) = 0$ and, consequently, $E'(t) \equiv 0$. In such cases, $E(t) = \text{const} = E(0)$.

Using Equation (68) and the initial conditions (66), we find that $E(0) = 0$. Therefore, $E(t) \equiv 0$ for both the first and the second boundary-value problems.

In the case of the third boundary-value problem we must proceed as follows. From the boundary condition (67) we find that

$$v_x(0, t) = \frac{\beta_1}{\alpha_1} v(0, t), \qquad v_x(l, t) = -\frac{\beta_2}{\alpha_2} v(l, t)$$

and if we substitute these values into (69), we obtain

$$E'(t) = -\frac{\beta_2}{\alpha_2} k(l) v(l, t) v_t(l, t) - \frac{\beta_1}{\alpha_1} k(0) v(0, t) v_t(0, t)$$

It is evident that this expression can also be written in the form

$$E'(t) = -\frac{\partial}{\partial t}\left[\frac{\beta_2}{2\alpha_2} k(l) v^2(l, t) + \frac{\beta_1}{2\alpha_1} k(0) v^2(0, t)\right]$$

Integrating this relationship between 0 and $t$, we find that

$$E(t)-E(0) = -\frac{\beta_2}{2\alpha_2}k(l)[v^2(l, t)-v^2(l, 0)]$$
$$-\frac{\beta_1}{2\alpha_1}k(0)[v^2(0, t)-v^2(0, 0)]$$

Since $E(0) = 0$, $v(l, 0) = v(0, 0) = 0$, it follows that

$$E(t) = -\frac{\beta_2}{2\alpha_2}k(l)v^2(l, t)-\frac{\beta_1}{2\alpha_1}k(0)v^2(0, t) \leqslant 0$$

However, it follows directly from the definition of $E(t)$ that $E(t) \geqslant 0$. Consequently, $E(t) \equiv 0$. Therefore, $E(t) \equiv 0$ for any solution of the problem defined by (65)–(67), i.e.

$$\int_0^l [kv_x^2+\rho v_t^2]\mathrm{d}x \equiv 0$$

Hence, it follows that

$$k(x)v_x^2(x, t)+\rho(x)v_t^2(x, t) \equiv 0$$

$$[k(x) > 0 \quad \text{and} \quad \rho(x) > 0], \qquad v_x(x, t) \equiv 0 \quad \text{and} \quad v_t(x, t) \equiv 0$$

Consequently, $v(x, t) \equiv$ const. Since $v(x, t)$ is continuous in a closed domain, we have

$$v(x, t) = v(x, 0) \equiv 0$$

which was to be proved.

*Remark* The requirement that the solution should be continuous in a closed domain is important since otherwise the solution would not be unique. In point of fact, if we add a function $\overline{u}(x, t)$ to the solution of, say, the first boundary-value problem, which is equal to a constant $C$ inside the domain $\{0 < x < l,\ t > 0\}$ and is zero on its boundary, we obtain a solution of this boundary-value problem for any value of $C$.

**4.4.2** Consider now the parabolic equation

$$\frac{\partial}{\partial x}[ku_x] = \rho u_t \tag{70}$$

The proof of the uniqueness of the solutions of boundary-value problems for parabolic equations is based on quite different ideas,

as we shall see below. To begin with, let us establish the following theorem.

*Theorem* Any solution $u(x, t)$ of Equation (70) which is continuous in the closed domain $\bar{D} \equiv \{0 \leqslant x \leqslant l,\ 0 \leqslant t \leqslant T\}$ assumes its maximum and minimum values either on the lower boundary of the domain $\bar{D}$ (for $t = 0$) or on end boundaries ($x = 0$, $x = l$).

*Proof* If $u(x, t) \equiv$ const, the validity of the theorem is obvious. Therefore, let us suppose that $u(x, t) \not\equiv$ const. To be specific, we shall prove the theorem for the maximum value. The proof for the minimum value follows the same course, except that $u(x, t)$ is replaced by $-u(x, t)$.

Let $M_\Gamma$ be the maximum value of $u(x, t)$ on the boundary $t = 0$, $x = 0$, $x = l$, and $M_D$ be the maximum value $u(x, t)$ in $\bar{D}$. It is required to show that $M_\Gamma = M_D$. Let us suppose that $M_D > M_\Gamma$. Consider the auxiliary function

$$v(x, t) = u(x, t)+\alpha(T-t)$$

where $\alpha > 0$ and $\alpha < (M_D - M_\Gamma)/2T$. The function $v(x, t)$ is continuous in $\bar{D}$ and, consequently, it reaches its maximum value in $\bar{D}$ at some point $(x_1, t_1)$. It is evident that $v(x_1, t_1) \geqslant M_D$ since $v(x, t) \geqslant u(x, t)$ in $\bar{D}$.

The point $(x_1, t_1)$ cannot lie on any of the three boundaries $t = 0$, $x = 0$, $x = l$. In fact,

$$|v(x, 0)| \leqslant |u(x, 0)|+\alpha T < M_\Gamma+\frac{1}{2}(M_D-M_\Gamma) < M_D$$

$$|v(0, t)| \leqslant |u(0, t)|+\alpha(T-t) \leqslant M_\Gamma+\frac{1}{2}(M_D-M_\Gamma) < M_D$$

$$|v(l, t)| \leqslant |u(l, t)|+\alpha(T-t) \leqslant M_\Gamma+\frac{1}{2}(M_D-M_\Gamma) < M_D$$

whereas $v(x_1, t_1) \geqslant M_D$.

Therefore, the point $(x_1, t_1)$ belongs to the domain $D \equiv \{0 < x < l,\ 0 < t \leqslant T\}$ and therefore the function $u(x, t)$ should satisfy Equation (70) in this domain. However,

$$u_x(x_1, t_1) = v_x(x_1, t_1) = 0, \qquad u_{xx}(x_1, t_1) = v_{xx}(x_1, t_1) \leqslant 0$$

$$u_t(x_1, t_1) = v_t(x_1, t_1)+\alpha > 0, \qquad (\text{or } v_t(x_1, t_1) \geqslant 0)$$

and, consequently, $u(x, t)$ does not satisfy Equation (70) at the point $(x_1, t_1)$. This contradiction means that the original hypothesis must be incorrect and, therefore, $M_D = M_\Gamma$.

This theorem is an expression of the obvious fact that heat (or diffusing matter) is transmitted only from points at the higher temperature (concentration) to points at lower temperature. Once the initial temperature (concentration) is specified on the boundary $t = 0$ of $D_1$, this process begins from $t = 0$ onwards. It is evident that the temperature at internal points cannot become higher than the temperature at $t = 0$. The same applies when the temperature is prescribed at $x = 0$ or $x = l$.

*Corollary 1* (Uniqueness of the solution of the first boundary-value problem). The solution of the first boundary-value problem

$$\frac{\partial}{\partial x}(k u_x)+f(x, t) = \rho u_t \tag{71}$$

$$u(0, t) = \mu_1(t), \qquad u(l, t) = \mu_2(t), \qquad u(x, 0) = \varphi(x)$$

which is continuous in the domain $\bar{D}_0$, $\bar{D}_0 \equiv \{0 \leqslant x \leqslant l,\ t \geqslant 0\}$ is unique.

Suppose that $u_1$ and $u_2$ are two solutions of this problem. The difference $u = u_1 - u_2$ is the solution of Equation (70) satisfying the following boundary and initial conditions

$$u(0, t) = u(l, t) = 0, \qquad u(x, 0) = 0$$

This solution is continuous in any domain $\bar{D} \equiv \{0 \leqslant x \leqslant l,$ $0 \leqslant t \leqslant T\}$, and therefore assumes maximum and minimum values on the boundaries of the domain ($t = 0$, $x = 0$, $x = l$). It is evident that these values are equal to zero. Consequently, $u(x, t) \equiv 0$ in $\bar{D}$, which was to be proved.

*Corollary 2* If the solutions $u_1(x, t)$ and $u_2(x, t)$ of Equation (71) are continuous in the domain $\bar{D}$, and on the boundaries of the domain ($t = 0$, $x = 0$, $x = l$) satisfy the inequality $u_1(x, t) < u_2(x, t)$, then the following inequality is satisfied throughout $\bar{D}$:

$$u_1(x, t) < u_2(x, t)$$

In fact, the function $u(x, t) = u_2 - u_1$ is a solution of (70); it is continuous in $\bar{D}$ and is positive on the boundaries $t = 0$, $x = 0$, $x = l$. Consequently, the maximum and minimum values of $u(x, t)$ are positive. It follows that $u(x, t) = u_2 - u_1 > 0$ throughout $\bar{D}$, which was to be proved.

*Corollary 3* (Continuity of the dependence of the solution of the first boundary-value problem on the boundary and initial values.) If in the boundary-value problems

$$\frac{\partial}{\partial x}(ku_x)+f(x, t) = \rho u_t$$

$$u(0, t) = \mu_1(t), \qquad u(l, t) = \mu_2(t), \qquad u(x, 0) = \varphi(x)$$

and

$$\frac{\partial}{\partial x}(k\bar{u}_x)+f(x, t) = \rho \bar{u}_t$$

$$\bar{u}(0, t) = \bar{\mu}_1(t), \qquad \bar{u}(l, t) = \bar{\mu}_2(t), \qquad \bar{u}(x, 0) = \bar{\varphi}(x)$$

the functions $\mu_1, \mu_2, \bar{\mu}_1, \bar{\mu}_2, \varphi, \bar{\varphi}$ satisfy the inequalities

$$| \mu_1(t)-\bar{\mu}_1(t) | < \varepsilon, \qquad | \mu_2(t)-\bar{\mu}_2(t) | < \varepsilon$$

for all values $t \geqslant 0$, and

$$| \varphi(x)-\bar{\varphi}(x) | < \varepsilon$$

at all points within the range $0 \leqslant x \leqslant l$, then for solutions $u(x, t)$ and $\bar{u}(x, t)$ which are continuous in $\bar{D}$, we have the following inequality everywhere in $\bar{D}$

$$| u(x, t)-\bar{u}(x, t) | < \varepsilon$$

This follows directly from Corollary 2.

## PROBLEMS

1. Determine the vibrations of a string $0 \leqslant x \leqslant l$ with rigidly fixed ends if, up to time $t = 0$, the string was in equilibrium under the action of a transverse force $F_0 = \text{const}$, applied at $x = x_0$ at right angles to the undisturbed position of the string, and at $t = 0$ the force $F_0$ is instantaneously removed.

2. Determine the vibrations of a string with rigidly fixed ends under the action of an impulse $P$, applied to the string at time $t = 0$ at the point $x = x_0$.

3. A rod with one end ($x = 0$) rigidly fixed is in equilibrium under the action of a longitudinal force $F_0 = \text{const}$ applied at the end $x = l$. At time $t = 0$ the force $F_0$ is instantaneously removed. Determine the subsequent vibrations of the rod.

4. One end of a rod ($x = l$) is fixed elastically and the other ($x = 0$) is given a longitudinal impulse $P$ at the initial time $t = 0$. Determine the subsequent vibrations of the rod.

5. Determine the temperature of a sphere or radius $R$ whose surface loses heat in accordance with Newton's law by convective heat transfer to the surrounding medium maintained at zero temperature. The initial temperature of the sphere is $f(r)$.

6. Investigate the cooling of a spherical shell $R_1 \leqslant r \leqslant R_2$ which loses heat from its two surfaces in accordance with Newton's law to an ambient medium at zero temperature. The initial temperature is $u(r, 0) = f(r)$, $R_1 < r < R_2$.

7. A closed spherical container of $0 \leqslant r \leqslant R$ contains diffusing material whose particles multiply at a rate proportional to the concentration. Determine the dimensions of the container (critical size) for which the process will diverge if (a) zero concentration is maintained at the surface of the container, (b) the wall of the container is impenetrable and (c) the wall is semipermeable.

8. Find the eigenvalues and eigenfunctions of a rectangular membrane with boundary conditions of types I, II and III. Show that in the case of a square there are two eigenfunctions to each eigenvalue.

9. Determine the eigenvalues and eigenfunctions of a rectangular parallelipiped for boundary conditions of types I, II and III.

10. Determine the natural frequencies of acoustic resonators in the form of (a) a rectangular parallelipiped and (b) a sphere.

11. Solve Problem 1 of Chapter 2.

12. Determine the longitudinal vibrations of a rod $0 \leqslant x \leqslant l$, one end of which is rigidly fixed and a force $F_0 =$ const is applied to the other at time $t = 0$.

13. Determine the temperature distribution in a rod $0 \leqslant x \leqslant l$, whose ends are maintained at a constant temperature ($u_1$ and $u_2$) and whose surface loses heat, in accordance with Newton's law, to the ambient medium maintained at a temperature $u_3 =$ const. The initial temperature can be assumed to be arbitrary.

14. Determine the temperature distribution in a rod $0 \leqslant x \leqslant l$ whose ends and surface lose heat in accordance with Newton's law to ambient media at constant temperature. The initial temperature is arbitrary.

15. The pressure and temperature of air in a cylinder $0 \leqslant x \leqslant l$ are kept at standard values. One end of the cylinder is open at time $t = 0$ and the other is kept closed. The concentration of a particular gas in the atmosphere is $u_0 =$ const. At time $t = 0$

the gas diffuses into the cylinder through the open end. Find the amount of gas $Q(t)$ which diffuses into the cylinder if the initial concentration in the cylinder is zero.

16. Solve Problem 15 by assuming that the diffusing gas propagates at a velocity proportional to its concentration.

17. Determine the electric field in a conductor $0 \leqslant x \leqslant l$ with negligible leakage and self-inductance, one end of which is insulated and a constant e.m.f. $E_0$ is applied to the other. The initial potential is $v_0 =$ const and the initial current is zero.

18. Determine the electric field in a conductor with negligible leakage and self-inductance if one end of it, $x = l$, is earthed and an e.m.f. $E_0$ is applied to the other through a lumped resistance $R_0$. Assume that the initial current and potential are both zero.

19. A conducting layer $0 \leqslant x \leqslant l$ is initially free of electromagnetic fields. At time $t = 0$ a magnetic field $H_0$, which is parallel to the layer, is applied to it. Determine the magnetic field in the layer for $t > 0$.

20. Determine the temperature of a rod $0 \leqslant x \leqslant l$ with thermally insulated surface if its initial temperature is zero, one end is maintained at zero temperature and the other is thermally insulated, given that a source of constant strength $Q$ is applied at time $t = 0$ at the point $x_0$, where $0 < x_0 < l$.

21. Determine the temperature of a uniform plate of zero initial temperature if a heat flux of constant density $q$ is applied to it through the plate $x = 0$ beginning with $t = 0$, and the face $x = l$ is maintained at a temperature $u_0 =$ const.

22. A constant current producing heat of density $Q =$ const is passed through a conductor in the form of a flat plate of thickness $l$ beginning with time $t = 0$. Determine the temperature of the plate for $t > 0$ if the faces of the plate lose heat in accordance with Newton's law to the surrounding medium. The temperature of the medium is $u_0 =$ const and the initial temperature of the plate is zero.

23. The initial temperature of a sphere $0 \leqslant r \leqslant R$ is $u_0 =$ const and its surface loses heat in accordance with Newton's law to the surrounding medium maintained at a temperature $u_1 =$ const. Determine the temperature of the sphere for $t > 0$.

24. A heat flux $Q$ (per unit time) enters a beam of semicircular cross-section through its flat surface and leaves through the remainder of the surface. Find the steady-state distribution of temperature over the cross-section of the beam, assuming that the heat fluxes entering and leaving are distributed with constant densities.

25. The pointer of an instrument is attached to the end of a rod of length $l$, fixed at $x = 0$. Determine the torsional vibrations of the rod if at the initial time $t = 0$ the pointer was deflected through an angle $\alpha$ and then released with zero initial velocity. The moment of inertia of the pointer with respect to the axis of rotation is $I_0$.

26. A string, $0 \leqslant x \leqslant l$, with rigidly fixed ends, is subjected at time $t = 0$ to a continuously distributed force of linear density (a) $\Phi = \Phi_0 \sin \omega t$, (b) $\Phi = \Phi_0 \cos \omega t$, where $\Phi = \text{const}$. Determine the vibrations of the string in each case.

27. Determine the vibrations of a string $0 \leqslant x \leqslant l$ with rigidly fixed ends subjected to a force $F = F_0 \sin \omega t$ (and $F_0 \cos \omega t$) applied at the point $x = 0$ at time $t = 0$ in the absence of resonance.

28. Determine the temperature of a rod $0 \leqslant x \leqslant l$ with its surface thermally insulated if at time $t = 0$ heat sources of density $\Phi(t) \sin(\pi/l)x$ appear in the rod. The initial temperature is zero and the ends are maintained at zero temperature.

29. A heat source moves with velocity $v_0 = \text{const}$ along a rod $0 \leqslant x \leqslant l$ whose surface loses heat to the surrounding medium in accordance with Newton's law. The temperature of this medium is zero and the amount of heat received by the rod from the source per unit time is $q = A e^{-ht}$, where $h$ is the heat-transfer coefficient in the heat-transfer equation for the rod $u_t = a^2 u_{xx} - hu$. Determine the temperature of the rod if its initial temperature is zero and the ends are maintained at zero temperature.

30. Determine the longitudinal vibrations of a rod $0 \leqslant x \leqslant l$ if the end $x = 0$ is rigidly fixed and a force $F = A \sin \omega t$ (and $A \cos \omega t$) is applied to the other end ($x = l$) at time $t = 0$, where $A = \text{const}$.

31. Determine the temperature distribution in a sphere $0 \leqslant r \leqslant R$ if its initial temperature is $u_0 = \text{const}$ and, beginning with $t = 0$, a constant heat flux of density $q = \text{const}$ enters the sphere through the surface.

32. A rod $0 \leqslant x \leqslant l$ with thermally insulated lateral surface and constant cross-section consists of two homogeneous rods $0 \leqslant x \leqslant x_0$, $x_0 \leqslant x \leqslant l$ with different physical properties. Determine the temperature distribution in the rod if the initial temperature is $f(x)$ and the ends are maintained at zero temperature.

33. Determine the temperature distribution in a homogeneous rod with its surface thermally insulated containing a thermal capacity $C_0$ at the point $x_0$. The initial temperature is arbitrary and the ends are maintained at zero temperature.

34. Determine the potential distribution in a conductor with negligible self-inductance and leakage if one end of it ($x = l$) is earthed through a lumped capacitance $C_0$ and a constant e.m.f. $E_0$ is applied to the other ($x = 0$). The initial potential and current are zero.

35. Find the solution of the first internal boundary-value problem in a circle of radius $R$ for the Laplace equation subject to the boundary conditions:

(a) $u(R, \varphi) = A\cos\varphi$, (b) $u(R, \varphi) = A+B\sin\varphi$, (c) $u|_{r=R} = Axy$,
(d) $u(R, \varphi) = A\sin^2\varphi+B\cos^2\varphi$.

36. Solve the second internal boundary-value problem in a circle of radius $R$ for the Laplace equation with boundary conditions:

(a) $\left.\dfrac{\partial u}{\partial n}\right|_C = A$, (b) $\left.\dfrac{\partial u}{\partial n}\right|_C = Ax$, (c) $\left.\dfrac{\partial u}{\partial n}\right|_C = A(x^2-y^2)$,

(d) $\left.\dfrac{\partial u}{\partial n}\right|_C = A\sin\varphi+B\sin^3\varphi$. Which of these problems is correctly formulated?

37. Solve the first internal boundary-value problem in a ring $R_1 < r < R_2$ for the Laplace equation subject to boundary conditions $u|_{r=R_1} = u_1$, $u|_{r=R_2} = u_2$. Use the solution of the problem to determine the capacitance of a cylindrical capacitor per unit length.

38. Determine the capacitance of a spherical capacitor filled with a dielectric of permittivity

$\varepsilon = \varepsilon_1$ for $a < r < c$ and $\varepsilon = \varepsilon_2$ for $c < r < b$.

39. Determine the electrostatic potential in a sphere of radius $R$, which is first charged to a potential $u_0$ and then placed in an infinite medium of permittivity $\varepsilon = \varepsilon_1$ for $R < r < c$ and $\varepsilon = \varepsilon_2$ for $r > c$. Consider the special cases $c = \infty$, $\varepsilon_2 = \infty$ and $\varepsilon_1 = \varepsilon_2 = \varepsilon$.

40. Find the solution of the internal boundary-value problems in a ring $R_1 < r < R_2$ for the equation $\nabla^2 u = A$ subject to the boundary conditions (a) $u|_{r=R} = u_1$, $u|_{r=R_2} = u_2$, (b) $u|_{r=R_1} = u_1$, $\left.\dfrac{\partial u}{\partial n}\right|_{r=R_2} = u_2$.

41. Find the solution of the boundary-value problem $\nabla^2 u = 1$, $u|_{r=R_1} = 0$, $u|_{r=R_2} = 0$ in a spherical layer $R_1 < r < R_2$.

42. Determine the potential distribution $u(x, y)$ inside a box of rectangular cross-section $-a < x < a$, $-b < y < b$ if two opposite faces of the box ($x = \pm a$) are at a potential $V_0$ and the others are earthed.

# 5
# The Method of Green's Functions for Parabolic Equations

In Chapter 4 we considered solutions of boundary-value problems for parabolic equations. In this chapter we shall discuss Cauchy's problem for the simplest parabolic equations for which the method of characteristics will no longer be suitable.

## 5.1 UNIQUENESS OF SOLUTIONS FOR PROPAGATION OF HEAT ALONG AN INFINITE STRAIGHT LINE

*Uniqueness theorem* The solution of Cauchy's problem

$$a^2 u_{xx} + f(x, t) = u_t \tag{1}$$

$$u(x, 0) = \varphi(x) \tag{2}$$

which is continuous and bounded in the closed domain $\bar{D}_1 \equiv \{-\infty < x + \infty, t \geqslant 0\}$ is unique.

*Proof* Let $u_1(x, t)$ and $u_2(x, t)$ be two solutions of the problem. By hypothesis, there exists a number $M$ such that $|u_1| \leqslant M$ and $|u_2| \leqslant M$ throughout $\bar{D}_1$. Consider the function $v(x, t) = u_2(x, t) - u_1(x, t)$ which is the solution of the problem

$$a^2 v_{xx} = v_t \tag{3}$$

$$v(x, 0) = 0 \tag{4}$$

and is continuous in $\bar{D}_1$, where $|v(x, t)| \leqslant 2M$ throughout $\bar{D}_1$.

Consider the domain $\overline{D}_b \equiv \{|x| \leqslant b, t \geqslant 0\}$ and the auxiliary function

$$w(x, t) = \frac{4M}{b^2}\left(\frac{x^2}{2} + a^2 t\right)$$

It is evident that $w(x, t)$ is a solution of (3) which is continuous in the domain $\overline{D}_b$. Moreover, on the boundaries of $\overline{D}_b$ we have $|v(x, t)| \leqslant w(x, t)$. In fact,

$$|v(x, 0)| = 0 \leqslant w(x, 0) = \frac{2M}{b^2} x^2$$

$$|v(\pm b, t)| \leqslant 2M \leqslant w(\pm b, t) = \frac{4M}{b^2}\left(\frac{b^2}{2} + a^2 t\right) = 2M + \frac{4M}{b^2} a^2 t$$

Therefore, Corollary 2 (Section 4.3) applies to the functions $v(x, t)$ and $w(x, t)$ in the domain $\overline{D}_b$. Accordingly, $|v(x, t)| \leqslant w(x, t)$ throughout $\overline{D}_b$. Consider now an arbitrary point $(x_1, t_1)$ of $\overline{D}_1$. For any sufficiently large $b$ this point will belong to $\overline{D}_b$. Consequently,

$$|v(x_1, t_1)| \leqslant \frac{4M}{b^2}\left(\frac{x_1^2}{2} + a^2 t_1\right)$$

If we take an arbitrary $\varepsilon > 0$ and sufficiently large $b$, we shall have

$$|v(x_1, t_1)| \leqslant \frac{4M}{b^2}\left(\frac{x_1^2}{2} + a^2 t_1\right) < \varepsilon$$

Consequently, $v(x_1, t_1) = 0$. Since the point $(x_1, t_1)$ is arbitrary, we have $v(x, t) = 0$, i.e. $u_2 \equiv u_1$ throughout $\overline{D}_1$.

## 5.2 THE FUNDAMENTAL SOLUTION (GREEN'S FUNCTION) ON A STRAIGHT LINE

**5.2.1** *Definition* The fundamental solution $G(x-x_0, t)$ of the simplest equation of heat transfer along an infinite straight line is defined as that solution of Cauchy's problem

$$a^2 u_{xx} = u_t \tag{5}$$

$$u(x, 0) = \delta(x-x_0) \tag{6}$$

which is continuous throughout $\overline{D}_1$ except for the point $(x_0, 0)$. The fundamental solution is frequently also referred to as Green's function.

Let us derive this function. Consider, to begin with, the following special Cauchy problem

$$a^2 u_{xx} = u_t \tag{7}$$

$$u(x, 0) = \varphi(x) = \eta(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geqslant 0 \end{cases} \tag{8}$$

We shall seek a solution of this problem among functions of the form $f(x/t^\alpha)$. Substituting the function $u = f(x/t^\alpha)$ into (7), we obtain

$$\frac{a^2}{t^{2\alpha}} f''(z) = -\frac{\alpha z}{t} f'(z), \quad \text{for} \quad z = \frac{x}{t^\alpha}$$

In order that this should be an identity in $z$, it is necessary that $\alpha = 1/2$. The equation for $f(z)$ is of the form

$$f''(z) + \frac{z}{2a^2} f'(z) = 0 \tag{9}$$

From the initial conditions (8) for $u(x, t)$ we have

$$f(-\infty) = 0, \qquad f(+\infty) = 1 \tag{10}$$

This formulates the problem for $f(z)$.

Integrating Equation (9) we obtain

$$\ln f'(z) = -\frac{z^2}{4a^2} + \ln C, \quad \text{for} \quad f'(z) = C e^{-z^2/4a^2}$$

and hence

$$f(z) = C \int_{-\infty}^{z} e^{-\xi^2/4a^2} \, d\xi = 2aC \int_{-\infty}^{z/2a} e^{-y^2} \, dy$$

This function satisfies the first condition in (10). From the second condition we obtain a relationship for $C$:

$$1 = 2aC \int_{-\infty}^{\infty} e^{-y^2} \, dy = 2aC \sqrt{\pi}$$

from which $C = 1/2a\sqrt{\pi}$. Therefore, the solution of the problem (7)–(9) is of the form

$$u(x, t) = f\left(\frac{x}{\sqrt{t}}\right) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{x}{\sqrt{4a^2 t}}} e^{-y^2} \, dy$$

It can also be written in the form

$$u(x, t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{0} e^{-y^2} dy + \frac{1}{\sqrt{\pi}} \int_{0}^{\frac{x}{\sqrt{4a^2t}}} e^{-y^2} dy$$

or

$$u(x, t) = \frac{1}{2}\left[1+\Phi\left(\frac{x}{\sqrt{4a^2t}}\right)\right]$$

where $\Phi(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-y^2} dy$ is the error function.

If the condition given by (8) is written in the form

$$u(x, 0) = u_0 \eta (x - x_0)$$

the solution of (7)–(8) will be the function

$$u(x, t) = \frac{u_0}{2}\left[1+\Phi\left(\frac{x-x_0}{\sqrt{4a^2t}}\right)\right]$$

If, on the other hand, $u(x, 0) = u_0[\eta(x-x_1)-\eta(x-x_2)]$,

$$u(x, t) = \frac{u_0}{2}\left[\Phi\left(\frac{x-x_1}{4a^2t}\right)-\Phi\left(\frac{x-x_2}{4a^2t}\right)\right] \tag{11}$$

Suppose now that an amount of heat $Q$ is liberated uniformly in the segment $[x_1, x_2]$ at the initial time $t = 0$. This is equivalent to specifying an initial temperature

$$u(x, 0) = \frac{Q}{c\rho(x_2-x_1)}[\eta(x-x_1)-\eta(x-x_2)]$$

In view of (11), the corresponding solution of Cauchy's problem is

$$u(x, t) = \frac{Q}{2c\rho} \frac{\Phi\left(\frac{x-x_1}{\sqrt{4a^2t}}\right)-\Phi\left(\frac{x-x_2}{\sqrt{4a^2t}}\right)}{x_2-x_1} \tag{12}$$

If we now allow the segment $[x_1, x_2]$ to contract to a point $x_0$, keeping $Q$ constant, the function (12) will tend to a limit given by

$$-\frac{Q}{c\rho}\frac{\partial}{\partial z}\Phi\left(\frac{x-z}{\sqrt{4a^2t}}\right)\bigg|_{z=x_0} = \frac{Q}{c\rho}\frac{1}{\sqrt{4\pi a^2t}}e^{-\frac{(x-x_0)^2}{4a^2t}}$$

Direct verification will show that the function

$$G(x-x_0, t) = \frac{1}{\sqrt{4\pi a^2 t}} e^{-\frac{(x-x_0)^2}{4a^2 t}} \tag{12a}$$

satisfies Equation (5). Moreover, $G(x-x_0, 0) = \delta(x-x_0)$, since by allowing $t$ to tend to zero in the sequence $\{t_n\}$, where $t_n = 1/4a^2 n$, we obtain a sequence of values of the function $G(x-x_0, t_n) = \sqrt{n/\pi}\, e^{-n(x-x_0)}$ which defines the $\delta$ function $\delta(x-x_0)$ (see Appendix). The continuity of the function $G(x-x_0, t)$ throughout $\bar{D}_1$, except for the point $(x_0, 0)$, is self-evident. Therefore, $G(x-x_0, t)$ is a solution of the problem (5)–(6) and, consequently, is the required fundamental solution. The function $G(x-\xi, t)$ gives the temperature of an infinite straight line (for example an infinite thin rod) for $t > 0$, in which an amount of heat $Q = c\rho$ is instantaneously released at $t = 0$ at the point $x = \xi$. When $Q \neq c\rho$, the temperature is $Q/c\rho\,[G(x-\xi, t)]$.

*Remark* If amounts of heat $Q_1$ and $Q_2$ are released instantaneously at $\xi_1$ and $\xi_2$ at the initial time, the temperature of the infinite straight line due to these sources will be

$$\frac{Q_1}{c\rho} G(x-\xi_1, t) + \frac{Q_2}{c\rho} G(x-\xi_2, t)$$

**5.2.2** *Definition* The fundamental solution of the simplest equation of heat transfer (Green's function) on a semi-infinite straight line subject to the boundary condition $u(0, t) = 0$ [or $u_x(0,t) = 0$] is defined as that solution of the problem

$$a^2 u_{xx} = u_t$$

$$u(x, 0) = \delta(x-x_0) - \delta(x+x_0)$$

[or, correspondingly, of the problem $a^2 u_{xx} = u_t$, $u(x, 0) = \delta(x-x_0) + \delta(x+x_0)$] which is continuous throughout the closed domain $\bar{D}_1$ except at the points $(-x_0, 0)$ and $(x_0, 0)$.

Using the above remark, we find the solution to be

$$G^*(x, x_0, t) = G(x-x_0, t) - G(x+x_0, t)$$

$$= \frac{1}{\sqrt{4\pi a^2 t}} \left[ e^{-\frac{(x-x_0)^2}{4a^2 t}} - e^{-\frac{(x+x_0)^2}{4a^2 t}} \right]$$

or, correspondingly,

$$G^{**}(x, x_0, t) = G(x-x_0, t)+G(x+x_0, t)$$
$$= \frac{1}{\sqrt{4\pi a^2 t}}\left[e^{-\frac{(x-x_0)^2}{4a^2t}}+e^{-\frac{(x-x_0)^2}{4a^2t}}\right]$$

## 5.3 PROPAGATION OF HEAT ALONG AN INFINITE STRAIGHT LINE

**5.3.1** We are now in a position to formulate the solution of Cauchy's problem. Consider, to begin with, the homogeneous equation

$$a^2 u_{xx} = u_t \tag{13}$$

$$u(x, 0) = \varphi(x) \tag{14}$$

We shall use the temperature interpretation of the problem.

Consider an element of length $d\xi$ containing the point $x = \xi$ on the line $t = 0$. The amount of heat liberated at time $t = 0$ in this element is $c\rho\varphi(\xi)d\xi$. This amount of heat can be referred to the point $\xi$. We thus have a point source which liberates the amount of heat $dQ = c\rho\varphi(\xi)d\xi$ at time $t = 0$ at the point $x = \xi$. The temperature distribution along the infinite straight line for $t > 0$, which results from this source, is

$$\frac{dQ}{c\rho}G(x-\xi, t) = \varphi(\xi)G(x-\xi, t)\,d\xi$$

This also applies to any other element of length $d\xi$ along the line $t = 0$. In view of the remark in Section 5.2, it is natural to suppose that the temperature due to the action of all such elements, i.e. the temperature due to the initial temperature distribution $u(x, 0) = \varphi(x)$ is given by

$$u(x, t) = \int_{-\infty}^{\infty} \varphi(\xi)G(x-\xi, t)\,d\xi \tag{15}$$

If this is correct, the function given by (15) will, in fact, be the solution of Cauchy's problem (13)–(14). To verify this statement, it is sufficient to show that this function satisfies Equation (13) for all $x$ in the range $-\infty < x < \infty$ and $t > 0$ and the initial condition given by (14).

To begin with, let us verify the condition given by (14). According to (15) and in view of the fact that $G(x-\xi, 0) = \delta(x-\xi)$, we have

$$u(x, 0) = \int_{-\infty}^{\infty} \varphi(\xi) G(x-\xi, 0)\, d\xi = \int_{-\infty}^{\infty} \varphi(\xi) \delta(x-\xi)\, d\xi = \varphi(x)$$

The last integral is equal to $\varphi(x)$ from the fundamental property of the $\delta$-function. It follows that the function (15) does, in fact, satisfy the condition given by (14).

To establish that the function (15) is a solution of (13), it is sufficient to show that this function can be differentiated with respect to $x$ (twice), and with respect to $t$, under the integration sign. In fact, if

$$u_{xx} = \int_{-\infty}^{\infty} \varphi(\xi) G_{xx}(x-\xi, t)\, d\xi \quad \text{and} \quad u_t = \int_{-\infty}^{\infty} \varphi(\xi) G_t(x-\xi, t)\, d\xi$$

then

$$a^2 u_{xx} - u_t = \int_{-\infty}^{\infty} \varphi(\xi) \{a^2 G_{xx} - G_t\}\, d\xi \equiv 0$$

since the function $G(x-\xi, t)$ is a solution of (13).

It is clearly sufficient to show that the integral (15) converges, whereas the integrals

$$\int_{-\infty}^{\infty} \varphi(\xi) G_t\, d\xi, \qquad \int_{-\infty}^{\infty} \varphi(\xi) G_x\, d\xi \quad \text{and} \quad \int_{-\infty}^{\infty} \varphi(\xi) G_{xx}\, d\xi \tag{15a}$$

converge uniformly in $D_\varepsilon \equiv \{-\infty < x < \infty, t \geqslant \varepsilon\}$ for arbitrary $\varepsilon > 0$.

We shall suppose that $\varphi(x)$ is bounded, i.e. $|\varphi(x)| \leqslant M$.

Let us substitute $\alpha = (\xi - x)/\sqrt{4a^2 t}$ in the integral (15). We then have (see Equation (12a))

$$u(x, t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \varphi(x + 2a\alpha\sqrt{t})\, e^{-\alpha^2}\, d\alpha$$

$$|u(x, t)| \leqslant \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} |\varphi(x + 2a\alpha\sqrt{t})| e^{-\alpha^2}\, d\alpha$$

$$\leqslant \frac{M}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\alpha^2}\, d\alpha = M \tag{16}$$

Therefore, the integral given by (15) does, in fact, converge (moreover, it converges uniformly) in the domain $\bar{D}_1$ and $|u| \leqslant M$. If we assume, in addition, that $\varphi(x)$ is continuous throughout, it will follow that the function (15) is continuous in the closed domain $\bar{D}_1$. (This involves the additional assumption that the function $\varphi(x)$ is bounded. If $\varphi(x)$ is piecewise-continuous, the function (15) is continuous throughout $\bar{D}_1$ except for the points on $t = 0$ at which $\varphi'(x)$ is discontinuous.)

*Remark* It follows from (16) that if the initial values $\varphi_1(x)$ and $\varphi_2(x)$ differ by less than $\varepsilon$, i.e. $|\varphi_1(x)-\varphi_2(x)| < \varepsilon$ for all $x$, then the corresponding solutions of Cauchy's problem, $u_1(x, t)$ and $u_2(x, t)$, will also differ from each other by less than $\varepsilon$, i.e.

$$|u_1(x, t)-u_2(x, t)| < \varepsilon$$

It follows that the solution of Cauchy's problem depends continuously on the initial values.

Consider now the first of the integrals in (15a)

$$\int_{-\infty}^{\infty} \varphi(\xi) G_t \, d\xi = -\int_{-\infty}^{\infty} \frac{\varphi(\xi)}{2t} G(x-\xi, t) \, d\xi + \int_{-\infty}^{\infty} \frac{\varphi(\xi)(x-\xi)^2}{4a^2t^2} G(x-\xi, t) \, d\xi$$

The substitution $\alpha = (\xi - x)/\sqrt{4a^2t}$ will reduce the first integral to

$$\int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi}\, t} \varphi(x+2a\alpha\sqrt{t}\,) \, e^{-\alpha^2} \, d\alpha$$

This integral converges uniformly in the domain $D_\varepsilon$ for arbitrary $\varepsilon > 0$, since the function $(M/2\sqrt{\pi}\varepsilon)e^{-\alpha^2}$ majorises the integrand and the integral of the former converges. The second integral in (15a) can be reduced to $\int_{-\infty}^{\infty} \frac{\alpha^2}{\sqrt{\pi}\, t} \varphi(x+2a\alpha\sqrt{t}) e^{-\alpha^2} \, d\alpha$ by the same substitution. This integral converges uniformly in the domain $D_\varepsilon$ for arbitrary $\varepsilon > 0$ since, in this domain, the function $(M/\varepsilon\sqrt{\pi})\alpha^2 e^{-\alpha^2}$ majorises the integrand and the integral of the former converges. The third integral in (15a) can be treated in a similar way. We have

thus shown that (15) does, in fact, give the solution of Cauchy's problem (13)–(14).

The solutions of the following problems can be obtained in a similar way:

1. $\quad a^2u_{xx} = u_t, \quad u(x, 0) = \varphi(x) \quad (0 \leqslant x < \infty), \quad u(0, t) = 0$

$$u(x, t) = \int_0^\infty \varphi(\xi) G^*(x, \xi; t)\, \mathrm{d}\xi \tag{17}$$

2. $\quad a^2u_{xx} = u_t, \quad u(x, 0) = 0 \quad (0 \leqslant x < x), \quad u_x(0, t) = 0$

$$u(x, t) = \int_0^\infty G^{**}(x, \xi; t)\varphi(\xi)\, \mathrm{d}\xi \tag{18}$$

The proof that the functions given by (17) and (18) are solutions of Problems 1 and 2 is quite similar to the proof given above.

*Remark 1* It follows from (15) that the heat propagates along the rod instantaneously. In fact, suppose that the initial temperature $\varphi(x)$ is positive over a finite segment $(x_1, x_2)$ of an infinite rod and is zero outside the segment. The temperature at an arbitrary point $x$ is then given by

$$u(x, t) = \int_{x_1}^{x_2} \varphi(\xi) G(x-\xi, t)\, \mathrm{d}\xi .$$

It is evident that this function is positive for any $x$ and any small $t > 0$. We have reached this conclusion because the physical assumptions which we have used in the formulation of Cauchy's problem were not strictly accurate [for example in writing Equation (13)].

*Remark 2* Formula (15) can be regarded as the convolution (see Appendix) of the fundamental solution

$$G(x, t) = \frac{1}{\sqrt{4\pi a^2 t}}\, \mathrm{e}^{-\frac{x^2}{4a^2t}}$$

with the initial function $\varphi(x)$, i.e.

$$u(x, t) = G(x, t) * \varphi(x)$$

If the initial function $\varphi(x)$ in this formula is taken to be an arbitrary finite generalised function, then $u(x, t)$ will also be a solution of Cauchy's problem.

**5.3.2** Consider a number of examples.

*Example 1* Solve the Cauchy problem

$$a^2 u_{xx} = u_t, \qquad u(x, 0) = \varphi(x) = \begin{cases} u_1, & x < 0 \\ u_2, & x \geqslant 0 \end{cases}$$

In accordance with (15)

$$u(x, t) = \int_{-\infty}^{\infty} \varphi(\xi) G(x-\xi, t)\, \mathrm{d}\xi$$

$$= u_1 \int_{-\infty}^{0} G(x-\xi, t)\, \mathrm{d}\xi + u_2 \int_{0}^{\infty} G(x-\xi, t)\, \mathrm{d}\xi$$

Substituting $\alpha = (x-\xi)/\sqrt{4a^2t}$ we obtain

$$u(x, t) = -\frac{u_1}{\sqrt{\pi}} \int_{\infty}^{\frac{x}{\sqrt{4a^2t}}} \mathrm{e}^{-\alpha^2}\, \mathrm{d}\alpha - \frac{u_2}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4a^2t}}}^{\infty} \mathrm{e}^{-\alpha^2}\, \mathrm{d}\alpha$$

$$= -\frac{u_1}{\sqrt{\pi}} \left( \int_{\infty}^{0} + \int_{0}^{\frac{x}{\sqrt{4a^2t}}} \right) - \frac{u_2}{\sqrt{\pi}} \left( \int_{\frac{x}{\sqrt{4a^2t}}}^{0} + \int_{0}^{-\infty} \right)$$

$$= \frac{u_1+u_2}{2} + \frac{u_2-u_1}{2} \Phi\left( \frac{x}{\sqrt{4a^2t}} \right)$$

*Example 2* Solve the Cauchy problem

$$a^2 u_{xx} = u_t, \qquad u(x, 0) = A\, \mathrm{e}^{-x^2}$$

From (15) we have

$$u(x, t) = A \int_{-\infty}^{\infty} \mathrm{e}^{-\xi^2} G(x-\xi, t)\, \mathrm{d}\xi$$

Substituting $\alpha = (\xi - x)/\sqrt{4a^2t}$, we obtain

$$u(x, t) = \frac{A}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(x+2a\alpha\sqrt{t})^2} e^{-\alpha^2} d\alpha$$

$$= \frac{A\, e^{-x^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-4xa\alpha\sqrt{t}-(4a^2t+1)\alpha^2} d\alpha$$

$$= \frac{A\, e^{-x^2}}{\sqrt{\pi}} e^{\frac{4x^2a^2t}{1+4a^2t}} \int_{-\infty}^{\infty} e^{-\left(\frac{2ax\sqrt{t}}{\sqrt{1+4a^2t}}+\sqrt{1+4a^2t}\,\alpha\right)^2} d\alpha$$

If we now substitute $2ax\sqrt{t}/\sqrt{1+4a^2t} + \sqrt{1+4a^2t}\ \alpha = \beta$ in the last integral, we obtain

$$u(x, t) = \frac{A}{\sqrt{\pi}} e^{\frac{-x^2}{1+4a^2t}} \frac{1}{\sqrt{1+4a^2t}} \int_{-\infty}^{\infty} e^{-\beta^2} d\beta = \frac{A}{\sqrt{1+4a^2t}} e^{\frac{-x^2}{1+4a^2t}}$$

**5.3.3** Consider now the inhomogeneous equation. The solution of Cauchy's problem

$$a^2u_{xx} + f(x, t) = u_t \tag{19}$$

$$u(x, 0) = \varphi(x) \tag{20}$$

will be sought in the form of the sum

$$u(x, t) = v(x, t) + w(x, t)$$

where the functions $v$ and $w$ are the solutions of the following problems:

$v$: $\qquad a^2v_{xx} = v_t, \qquad v(x, 0) = \varphi(x)$

$w$: $\qquad a^2w_{xx} + f(x, t) = w_t \qquad (19)$

$$w(x, 0) = 0 \tag{21}$$

The function $v(x, t)$ is given by (15).

To obtain the solution of the problem defined by (19), (21), we shall use again the temperature interpretation of Equation (19). In this equation $c\rho f(x, t)$ is the density of heat sources per unit time. Consequently, the element of length $d\xi$ containing the point $\xi$ will liberate the amount of heat $dQ = c\rho f(\xi, \tau) d\xi\, d\tau$ in the time interval $(\tau, \tau + d\tau)$. If this amount of heat is considered to be released

at the point $\xi$ instantaneously at time $\tau$, the temperature due to this source is

$$\frac{dQ}{c\rho} G(x-\xi, t-\tau) = f(\xi, \tau) G(x-\xi, t-\tau) \, d\xi \, d\tau$$

In view of the remark in Section 5.2, it is natural to suppose that the temperature distribution due to all such sources distributed along the line in the time interval between 0 and $t$ will be given by

$$w(x, t) = \int_0^t \int_{-\infty}^{\infty} f(\xi, \tau) G(x-\xi, t-\tau) \, d\xi \, d\tau \tag{22}$$

This formula is, in fact, the solution of the problem (19), (21). The condition given by (21) will, of course, be satisfied. To verify that the function given by (22) does, in fact, satisfy Equation (19) can be confirmed by analogy with the analysis given in connection with Equation (15).

If the heat source acts only at the point $\xi_0$, but is a function of time, the function $f(x, t)$ in (19) is of the form

$$f(x, t) = f(t)\delta(x-\xi_0)$$

The solution of the problem (19), (21) is then of the form

$$\begin{aligned} w(x, t) &= \int_0^t \int_{-\infty}^{\infty} \delta(\xi-\xi_0) f(\tau) G(x-\xi, t-\tau) \, d\xi \, d\tau \\ &= \int_0^t f(\tau) G(x-\xi_0, t-\tau) \, d\tau \end{aligned}$$

where we have used the properties of the $\delta$-function.

*Remark 3* The solution of Cauchy's problem for the inhomogeneous equation which is initially zero,

$$a^2 u_{xx} + f(x, t) = u_t, \qquad u(x, 0) = 0$$

can also be written as the convolution (two variables!) of $G(x, t)$ with $f(x, t)$:

$$u(x, t) = G(x, t) * f(x, t) = \int_0^t \int_{-\infty}^{\infty} G(x-\xi, t-\tau) f(\xi, \tau) \, d\xi \, d\tau$$

*Remark 4* The solution of the problem (19), (21) on the semi-infinite line with the boundary condition $w(0, t) = 0$ [or $w_x(0, t) = 0$] can be obtained in a similar way and is given by

$$w(x, t) = \int_0^t \int_0^\infty f(\xi, \tau) G^*(x, \xi, t-\tau)\, d\xi\, d\tau$$

(or $w = \int_0^t \int_0^\infty f G^{**} d\xi\, d\tau$).

## 5.4 PROPAGATION OF HEAT IN THREE-DIMENSIONAL (TWO-DIMENSIONAL) SPACE

Consider now the Cauchy problem for the equation of heat conduction in two- and three-dimensional space. To begin with, consider the homogeneous equation

$$a^2 \nabla^2 u = u_t \tag{23}$$

*Definition* The fundamental solution $G(M, M_0, t)$ of (23) is defined to be the solution which

1. satisfies the initial condition

$$u(M, 0) = \delta(M, M_0) \tag{24}$$

and

2. is continuous throughout the closed domain $\bar{D}_3 \equiv \{-\infty < x, y, z < \infty, t \geqslant 0\}$, except for the point $(x_0, y_0, z_0, 0)$. $x, y, z$ and $x_0, y_0, z_0$ are the coordinates of $M$ and $M_0$, respectively, and $\delta(M, M_0)$ is the $\delta$-function with the singularity at the point $M_0$. To determine $G(M, M_0, t)$ let us establish the following lemma.

*Lemma* If in the Cauchy problem

$$a^2 \nabla^2 u = u_t, \qquad u(M, 0) = \varphi(M)$$

the initial function $\varphi(M)$ is

$$\varphi(M) = \varphi_1(x)\varphi_2(y)\varphi_3(z)$$

the solution of the problem will be the function

$$u(M, t) = u_1(x, t) u_2(y, t) u_3(z, t)$$

where $u_1(x, t)$, $u_2(y, t)$, $u_3(z, t)$ are the solutions of the corresponding one-dimensional problems:

$$a^2 u_{1xx} = u_{1t}, \qquad u_1(x, 0) = \varphi_1(x)$$

and so on.

*Proof* By hypothesis

$$a^2\nabla^2(u_1u_2u_3) = u_2u_3a^2u_{1xx}+u_1u_3a^2u_{2yy}+u_1u_2a^2u_{3zz}$$
$$\equiv u_2u_3u_{1t}+u_1u_3u_{2t}+u_1u_2u_{3t} \equiv (u_1u_2u_3)_t$$

and therefore $u = u_1u_2u_3$ satisfies the given equation and, clearly, the initial condition also.

We note that $\delta(M, M_0) = \delta(x-x_0)\delta(y-y_0)\delta(z-z_0)$. Applying the above lemma to the problem (23)–(24), we obtain the required fundamental solution

$$G(M, M_0, t) = G(x-x_0, t)G(y-y_0, t)G(z-z_0, t)$$

Using the formulae for the one-dimensional fundamental solutions $G(x-x_0, t)$, etc., we obtain

$$G(M, M_0, t) = \left(\frac{1}{\sqrt{4\pi a^2 t}}\right)^3 e^{-\frac{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}{4a^2t}}$$

for the three-dimensional space, and

$$G(M, M_0, t) = \left(\frac{1}{\sqrt{4\pi a^2 t}}\right)^2 e^{-\frac{(x-x_0)^2+(y-y_0)^2}{4a^2t}}$$

for the two-dimensional space.

If an amount of heat equal to $Q$ is liberated at the point $M_0$ at the initial time $t = 0$, the temperature at an arbitrary point $M$ due to this source for $t > 0$ will be $Q/c\rho[G(M, M_0, t)]$. Using this fact, it is easy to construct the solution of the Cauchy problem for the homogeneous equation

$$a^2\nabla^2 u = u_t, \qquad u(x, y, z, 0) = \varphi(x, y, z)$$

The solution will be the function

$$u(x, y, z, t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(\xi, \eta, \zeta)G(x, y, z, \xi, \eta, \zeta, t)\,d\xi\,d\eta\,d\zeta \tag{25}$$

The solution of the problem

$$a^2\nabla^2 u+f(x, y, z, t) = u_t, \qquad u(x, y, z, 0) = 0$$

will be the function

$$u(x, y, z, t) = \int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi, \eta, \zeta, \tau)G(x, y, z, \xi, \eta, \zeta; t-\tau)\,d\xi\,d\eta\,d\zeta\,d\tau \tag{26}$$

The solution of the Cauchy problem for the inhomogeneous equation

$$a^2\nabla^2 u+f(M, t) = u_t, \qquad u(M, 0) = \varphi(M)$$

is equal to the sum of the functions (25) and (26).

The proof of these statements is almost the same as for the one-dimensional case and will not be reproduced here.

*Remark 5* The solution of the Cauchy problem

$$a^2\nabla^2 u = u_t, \qquad u(M, 0) = \varphi(x, y, z)$$

can also be written in the form of the convolution (in three variables!) of the fundamental solution

$$G(x, y, z, t) = \left(\frac{1}{\sqrt{4\pi a^2 t}}\right)^3 e^{-\frac{x^2+y^2+z^2}{4a^2t}}$$

and the initial function $\varphi(x, y, z)$:

$$\begin{aligned} u(x, y, z, t) &= G(x, y, z, t) * \varphi(x, y, z) \\ &= \iiint\limits_{-\infty}^{\infty} G(x-\xi, y-\eta, z-\zeta, t)\varphi(\xi, \eta, \zeta)\, d\xi\, d\eta\, d\zeta \end{aligned}$$

We have illustrated the application of the method of source functions to the solution of problems in infinite space and half-space. It is also possible to use this method to solve boundary-value problems in bounded (in the space variables) regions. For example, if we define the fundamental solution of the first boundary-value problem as the solution of the problem

$$a^2u_{xx} = u_t$$

$$u(0, t) = 0 = u(l, t), \qquad u(x, 0) = \delta(x-x_0)$$

which is continuous throughout the domain $\bar{D}_l \equiv \{0 \leqslant x \leqslant l, t \geqslant 0\}$ except for the point $(x_0, 0)$ [we shall denote this solution by $\bar{G}(x, x_0, t)$], the solution of the problem

$$a^2u_{xx} = u_t$$

$$u(0, t) = u(l, t) = 0, \qquad u(x, 0) = \varphi(x)$$

can be written in the form

$$u(x, t) = \int_0^l \varphi(\xi)\bar{G}(x, \xi; t)\, d\xi$$

We note that if we solve this problem by the method of separation of variables, we find that

$$\bar{G}(x, x_0, t) = \frac{2}{l} \sum_{n=1}^{\infty} \sin \frac{\pi n}{l} x \sin \frac{\pi n}{l} x_0 \sin \frac{\pi a n t}{l}$$

The situation is quite similar for the second and third boundary-value problems. However, this method is not ideal for bounded regions and is not usually employed.

## PROBLEMS

1. The initial current and voltage in a semi-infinite uniform conductor $0 \leqslant x \leqslant \infty$ are both zero. The self-inductance per unit length of the conductor is negligible. A constant e.m.f. $E_0$ is applied to one end of the conductor at time $t = 0$. Determine the voltage in the conductor for $t > 0$.

2. Determine the temperature distribution in infinite space given that $Q$ uniformly distributed units of heat were released instantaneously at time $t = 0$ on a spherical surface of radius $r_0$.

3. Find the concentration of diffusing matter in infinite space if the initial concentration of the material is $u|_{t=0} = u_0 = \text{const}$ for $0 \leqslant r < R$ and $u|_{t=0} = 0$ for $r > R$.

4. Solve Problem 3 for the half-space $z > 0$ assuming that $z_0 < R$ where $(0, 0, z_0)$ are the coordinates of the centre of the sphere in which the initial concentration is $u_0$. Consider the case where (a) the plane $z = 0$ is impermeable to the diffusing material and (b) zero concentration is maintained on the plane $z = 0$.

5. Determine the temperature distribution in a semi-infinite rod $0 \leqslant x < \infty$ with thermally insulated ends and lateral surface, due to heat sources of density $Q(t)$ on the segment $(a, b)$ $(0 < a < b)$ beginning with time $t = t_0$.

6. Use the source function found in Problem 2 to solve the boundary-value problem

$$a^2\left(u_{rr} + \frac{2}{r} u_r\right) + f(r, t) = u_t, \qquad 0 < r, t < \infty$$

$$u(r, 0) = \varphi(r), \qquad |u| < \infty, \qquad r^2 = x^2 + y^2 + z^2$$

7. Find the temperature distribution in infinite space using the fact that $Q$ uniformly distributed units of heat are instantaneously

released at $t = 0$ on each unit of length of an infinite cylindrical surface of radius $r_0$.

8. Use the source function found in Problem 7 to solve the following boundary-value problem:

$$a^2\left(u_{rr}+\frac{1}{r}u_r\right)+f(r, t) = u_t, \qquad 0 < r, t < \infty$$

$$u(r, 0) = \varphi(r), \qquad |u| < \infty, \qquad r^2 = x^2+y^2$$

9. Determine Green's function for a point source on an infinite line for the equation

$$a^2u_{xx}-hu = u_t$$

# 6

# The Method of Green's Functions for Elliptical Equations

In this and in the following chapter we shall be concerned with the principal methods for the solution of boundary-value problems for elliptical equations of the form $\nabla^2 u = f(M)$, and with the uniqueness of these solutions.

## 6.1 GREEN'S FORMULA. SIMPLEST PROPERTIES OF HARMONIC FUNCTIONS

All the results which are derived in this chapter can be deduced from a small number of formulae and relationships. These will first be derived.

**6.1.1** Suppose that the functions $u(M)$ and $v(M)$ have the following properties:

1. These functions and their first-order derivatives are continuous throughout a closed domain $\bar{D}$, bounded by the surface $S$, except, possibly, for a finite number of points.

2. The functions and their first-order derivatives are integrable in $D$.

3. The functions have second-order partial derivatives which are integrable in $D$.

Subject to these conditions, we have

$$R[u, v] = -\int_D vL[u]\,\mathrm{d}\tau$$

$$= \int_D k(\nabla u \cdot \nabla v)\,\mathrm{d}\tau + \int_D quv\,\mathrm{d}\tau - \int_S kv\frac{\partial u}{\partial n}\,\mathrm{d}\tau \qquad (1)$$

where $L[u] \equiv \operatorname{div}(k\,\nabla u)-qu$. Subtracting $R[u,v]$ from $R[v,u]$, we obtain Green's formula

$$\int\limits_D \{vL[u]-uL[v]\}\,\mathrm{d}\tau = \int\limits_S k\left(v\frac{\partial u}{\partial n}-u\frac{\partial v}{\partial n}\right)\mathrm{d}\sigma \tag{2}$$

For the one-dimensional case this assumes the form

$$\int\limits_0^l \{vL[u]-uL[v]\}\,\mathrm{d}x = k\left(v\frac{\mathrm{d}u}{\mathrm{d}x}-u\frac{\mathrm{d}v}{\mathrm{d}x}\right)\Big|_0^l \tag{2$_1$}$$

*Corollary* Suppose that $L[u] \equiv \operatorname{div}(k\nabla u)$. If we set $v \equiv 1$ for this operator in (2) and take $u(M)$ to be the solution of the equation $L[u] = f(M)$ which, together with the first-order partial derivatives, is continuous in the domain $\overline{D} = D+S$, then

$$\int\limits_S k\frac{\partial u}{\partial n}\,\mathrm{d}\sigma = \int\limits_D f(M)\,\mathrm{d}\tau \tag{3}$$

For a doubly-connected region $D'$, bounded by two concentric spheres $S_R$ and $S_{R_1} (R_1 < R)$, centred on the point $M_0$, Equation (2) can be written in the form

$$\int\limits_{D'}\{vL[u]-uL[v]\}\,\mathrm{d}\tau = \int\limits_{S_R} k\left(v\frac{\partial u}{\partial r}-u\frac{\partial v}{\partial r}\right)\mathrm{d}\sigma - \int\limits_{S_{R_1}} k\left(v\frac{\partial u}{\partial r}-u\frac{\partial v}{\partial r}\right)\mathrm{d}\sigma \tag{4}$$

The minus sign in front of the integral over $S_{R_1}$ has appeared because on this surface $\partial/\partial n = -\partial/\partial r$.

Suppose that $L[u] \equiv \nabla^2 u$. The continuous solutions of the Laplace equation $\nabla^2 u = 0$ are called harmonic functions. Direct verification will show that in three-dimensional space the function $1/r$ is harmonic everywhere except for the point $r = 0$. In the two-dimensional case, the function $\ln\,(1/r)$ is harmonic everywhere except for the point $r = 0$.

Using (3) for functions which are harmonic in a domain $D$ and are continuous together with their first-order partial derivatives in $D+S$, we obtain

$$\int\limits_S \frac{\partial u}{\partial n}\,\mathrm{d}\sigma = 0 \tag{5}$$

**6.1.2** *Mean value theorem* If the function $u(M)$ is harmonic in a spherical domain $D_R$ and is continuous, together with its first-order partial derivatives, in $\bar{D}_R = D_R + S_R$, then its value at the centre $M_0$ of $D_R$ is equal to the arithmetic mean of its values on the sphere $S_R$, i.e.

$$u(M_0) = \frac{1}{4\pi R^2} \int_{S_R} u(M)\, d\sigma \tag{6}$$

*Proof* Let us substitute $L[u] \equiv \nabla^2 u$ in (4), so that

$$v = \frac{1}{r_{M_0M}} \left(r_{M_0M} = \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}\right)$$

and take $u(M)$ to be a function which is harmonic in $D_R$, bounded by $S_R$ and continuous together with $u_x$, $u_y$, $u_z$ in $D_R + S_R$. Subject to these conditions, the integral over $D'(D' \subset D_R)$ is equal to zero. The integrals over $S_R$ and $S_{R_1}$ of the product $v\, \partial u/\partial r$ are also zero in view of (5) (the function $v$ is equal to $1/R$ and $1/R_1$ on these surfaces, respectively, and $k \equiv 1$). We therefore have

$$\int_{S_R} u \frac{\partial}{\partial r}\left(\frac{1}{r}\right) d\sigma - \int_{S_{R_1}} u \frac{\partial}{\partial r}\left(\frac{1}{r}\right) d\sigma = 0$$

Evaluating the derivatives and applying to the last integral the mean value theorem for integrals, we obtain

$$\frac{1}{R^2} \int_{S_R} u(M)\, d\sigma = \frac{1}{R_1^2} 4\pi R_1^2 u(M^*), \qquad M^* \in S_{R_1}$$

If we now go to the limit $R_1 \to 0$ we obtain Equation (6).

We have considered harmonic functions in three-dimensional space. In two-dimensional space (plane) the mean value theorem is given by

$$u(M_0) = \frac{1}{2\pi R} \int_{C_R} u(M)\, ds \tag{7}$$

where $C_R$ is a circle centred on $M_0$. To derive this formula we must take $v = \ln(1/r_{M_0M})$ in the relationship analogous to (4).

## 6.2 UNIQUENESS OF SOLUTIONS OF BOUNDARY-VALUE PROBLEMS

In this section we shall consider the uniqueness of the solutions of the first and second boundary-value problems. It will be necessary to distinguish between internal and external boundary-value

problems. The formulation of the former problems is given in Chapter 2. The first external boundary-value problem involves the determination of the function which satisfies the equation $L[u] = f(M)$ at points lying outside the closed surface $S$, and assumes given values on $S$, i.e. $u|_S = \varphi(M)$. The second boundary-value problem is formulated in a similar way.

**6.2.1** Consider to begin with the uniqueness of the solutions of internal boundary-value problems.

*Theorem* (*maximum and minimum values*) A function $u(M)$ which is harmonic in a finite domain $D$ bounded by the closed surface $S$ and is continuous in $\bar{D} = D+S$ reaches its maximum and minimum values on the boundary $S$.

*Proof* Let $H_S$ represent the maximum value of $u$ on $S$ and $H_D$ the maximum value of $u$ in $\bar{D}$. It is required to show that $H_D = H_S$. Let us suppose that this is not so. We then have $H_D > H_S$ and $u(M_0) = H_D$ at some point $M_0 (M_0 \in D)$.

Consider the auxiliary function

$$v(M) = u(M) + \frac{H_D - H_S}{2d^2}[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]$$

where $d$ is the diameter of $D$, i.e. the upper limit of the distance between points in $D$; $(x_0, y_0, z_0)$ and $(x, y, z)$ are the coordinates of the points $M_0$ and $M$, respectively. It is evident that for all points $M \in D$

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2 < d^2$$

and $v(M_0) = u(M_0) = H_D$. On the other hand, at a point $M$ on $S$ we have

$$v(M) \leqslant H_S + \frac{H_D - H_S}{2} = \frac{H_D + H_S}{2} < H_D$$

Consequently, the function $v(M)$, which is continuous in $\bar{D}$, should reach its maximum value at some internal point $M_1$ of $D$. At this point we should have $\nabla^2 v \leqslant 0$ since, at the point where the maximum is reached, none of the derivatives $v_{xx}, v_{yy}, v_{zz}$ can be positive. On the other hand,

$$\nabla^2 v = \nabla^2 u + 3\frac{H_D - H_S}{d^2} = 3\frac{H_D - H_S}{d^2} > 0$$

This contradiction shows that the original hypothesis, namely, $H_D > H_S$, was incorrect. Consequently, $H_D = H_S$. By applying this result to the function $-u$, we obtain the proof of the theorem for the minimum value.

The uniqueness of the solution of the first internal boundary-value problem for the equation $\nabla^2 u = f(M)$ follows directly from this theorem.

*Uniqueness theorem* The solution of the first internal boundary-value problem

$$\nabla^2 u = f(M), \qquad u|_S = \varphi(M)$$

which is continuous in the closed domain $\bar{D} = D + S$, is unique.

*Proof* Suppose that the two functions $u_1$ and $u_2$ are the solutions of this problem. Their difference $u = u_1 - u_2$ is a function which is harmonic in $D$, continuous in $\bar{D}$ and zero on $S$. In view of the above theorem, the maximum and minimum values of $u$ are zero and, consequently, $u = u_1 - u_2 \equiv 0$ throughout $D$.

It is readily shown that the solutions of the first internal boundary-value problem depend continuously on the boundary values for the equation $\nabla^2 u = f(M)$.

*Theorem* Let $u_1(M)$ and $u_2(M)$ be the solutions of the first internal boundary-value problem for the equation $\nabla^2 u = f(M)$, which are continuous in $\bar{D}$ and assume the values $\varphi_1(M)$ and $\varphi_2(M)$ on the boundary $S$ of the domain $D$. If then the inequality $|\varphi_1 - \varphi_2| < \varepsilon$ is satisfied everywhere on $S$, the inequality

$$|u_1(M) - u_2(M)| < \varepsilon$$

is satisfied throughout $D$.

*Proof* The function $u = u_1 - u_2$ is harmonic in $D$, continuous in $\bar{D}$ and $u|_S = \varphi_1 - \varphi_2$. Since $-\varepsilon < \varphi_1 - \varphi_2 < \varepsilon$, it follows from the previous theorem that the maximum and minimum values of $u(M)$ lie between $-\varepsilon$ and $\varepsilon$. Consequently $|u| < \varepsilon$, i.e. $|u_1(M) - u_2(M)| < \varepsilon$. The following theorem is also valid.

*Theorem* If a sequence of functions $u_1, u_2, \ldots, u_n, \ldots$, which are continuous in a closed and bounded domain $\bar{D}$ and are harmonic in $D$, converges uniformly on the boundary of the domain, it will

also converge uniformly in $\bar{D}$. The reader is recommended to prove this theorem using the Cauchy criterion for the convergence of the sequence.

**6.2.2** *Theorem* All the solutions of the second internal boundary-value problem

$$L[u] \equiv \operatorname{div}(k\nabla u) - qu = f(M), \quad \left.\frac{\partial u}{\partial n}\right|_S = \varphi(M) \quad (k > 0, q \geqslant 0)$$

which are continuous together with their first-order partial derivatives in a closed domain $\bar{D} = D+S$, can differ from each other only by a constant, i.e. for any two solutions $u_1$ and $u_2$, we have the identity $u_1 - u_2 \equiv \text{const}$. For different pairs of solutions these constants will, in general, be different.

*Proof* Let $u_1$ and $u_2$ be two solutions of the problem. The function $w = u_1 - u_2$ is a solution of the problem

$$L[w] = 0, \quad \left.\frac{\partial w}{\partial n}\right|_S = 0$$

Using (1) for $u = w$ and $v = w$ we obtain

$$\int_D k(\nabla w)^2 \, d\tau + \int_D qw^2 \, d\tau - \int_S kw \frac{\partial w}{\partial n} \, d\sigma = -\int_D wL[w] \, d\tau = 0$$

In view of the boundary conditions, the integral over $S$ is zero and, therefore,

$$\int_D \{k(\nabla w)^2 + qw^2\} \, d\tau = 0$$

whence $k(\nabla w)^2 + qw^2 \equiv 0$. Consequently, $k(\nabla w)^2 \equiv 0$ and $qw^2 \equiv 0$. If $q \not\equiv 0$, then $w = u_1 - u_2 \equiv 0$. If $q \equiv 0$, then from the identity $k(\nabla w)^2 \equiv 0$ it follows that $\nabla w \equiv 0$ since $k > 0$ and, hence, $w = u_1 - u_2 = \text{const}$. This proves the theorem.

*Remark* For arbitrary functions $\varphi(M)$ and $f(M)$ (even if they are continuous), the second boundary-value problem cannot have a solution. In fact, let $L[u] \equiv \operatorname{div}(k\nabla u)$. The relationship given by (3) must then be satisfied for the solution $u(M)$ of the second boundary-value problem, which is continuous in $\bar{D} = D+S$, together with its first-order partial derivatives. Since $\left.\frac{\partial u}{\partial n}\right|_S = \varphi(M)$, we have

$$\int_D f(M) \, d\tau = \int_S k \frac{\partial u}{\partial n} \, d\sigma = \int_S k\varphi(M) \, d\sigma$$

Therefore, the functions $f(M)$ and $\varphi(M)$ must be related by

$$\int_D f(M)\,\mathrm{d}\tau = \int_S k\varphi(M)\,\mathrm{d}\sigma \tag{8}$$

In particular, if $f(M) = 0$, the function $\varphi(M)$ must satisfy the condition

$$\int_S k\varphi\,\mathrm{d}\sigma = 0 \tag{9}$$

The physical significance of (8) and (9) is readily understood if $u(M)$ is interpreted as the steady-state temperature distribution and $k(M)$ as the thermal conductivity. The relationship given by (8) then expresses the following self-evident fact: in order to obtain a steady-state solution it is necessary that the amount of heat released by internal sources in the domain $D$ during time $\Delta t$ be equal to the total heat flux which has passed through the boundary $S$ of the domain during the same interval of time.

The uniqueness of the solution of the third internal boundary-value problem which is continuous in $\bar{D}$, together with its first-order derivatives can be proved in a similar way. The reader is recommended to establish the proof for himself.

**6.2.3** To ensure the uniqueness of solutions of external boundary-value problems, these solutions must satisfy additional conditions with regard to their behaviour at infinity. In fact, if we seek the solution of external boundary-value problems, these solutions must satisfy additional conditions with regard to their behaviour at infinity. For instance, if we seek the solution of the first external boundary-value problem for $r > R$, subject to the boundary condition $u|_{r=R} = C$, where $C$ is a constant, the solutions will be $u_1 \equiv C$, $u_2 = Cr/R$ and $u = Au_1 + Bu_2$, where $A$ and $B$ are arbitrary constants such that $A+B = 1$.

Let $D_1$ denote the domain of points lying outside the closed surface $S$. In three-dimensional space we then have the following theorem.

*Theorem* The solution $u(M)$ of the first external boundary-value problem for the equation $\nabla^2 u = f(M)$, which is continuous in the closed domain $\bar{D}_1 = D_1 + S$ and tends uniformly to zero as $M$ tends to infinity, is unique.

*Proof* Let $u_1$ and $u_2$ be two solutions of the problem. The function $u = u_1 - u_2$ is harmonic in $D_1$, continuous in $\bar{D}_1$ and $u|_S = 0$.

Let us construct a sphere $S_R$ centred on some given point, $M_0$, of $D$, which is bounded by the surface $S$ and is large enough

for $S$ to lie entirely in $D_1$ (Fig. 6.1). The function $u$ is harmonic in the domain $D_2$, bounded by the surfaces $S_R$ and $S$, and is continuous in $\bar{D}_2$. Consequently, in accordance with the theorem on the maximum and minimum values, the function $u$ assumes maximum and minimum values at points on the surfaces $S_R$ and $S$.

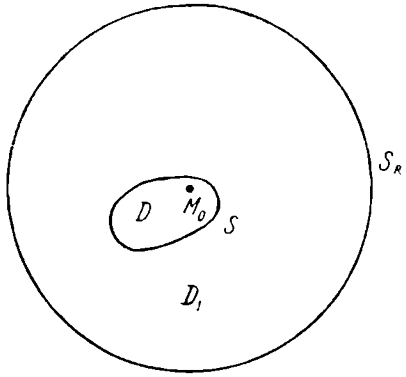



Fig. 6.1

Let us choose an arbitrary $\varepsilon > 0$ and take $R$ to be so large that $S_R$ has $|u| < \varepsilon$. This is possible because $u(M)$ tends uniformly to zero as $M$ tends to infinity. Since $u|_S = 0$, the theorem just mentioned will ensure that $|u| < \varepsilon$ everywhere in $D_2$. Since $\varepsilon$ is arbitrary, this means that $u = u_1 - u_2 \equiv 0$. This proves the theorem.

In the two-dimensional case, instead of the condition that $u(M)$ must tend uniformly to zero as $M$ tends to infinity, it is necessary that the solution be bounded in $D_1$. The proof of the uniqueness theorem for this case will not be reproduced here.

In the case of the second external boundary-value problem in three-dimensional space, the solution must both tend uniformly to zero at infinity and its second-order partial derivatives must follow a certain prescribed behaviour at infinity. Thus, if the required solution tends uniformly to zero at infinity, whilst the partial derivatives $u_x$, $u_y$, $u_z$ tend to zero (as $M$ tends to infinity) as $A/r^2$, the solution of the second external boundary-value problem is unique.

## 6.3 THE METHOD OF GREEN'S FUNCTIONS

We shall now consider methods for solving boundary-value problems for equations of the elliptic type:

$$L[u] = f(M) \tag{10}$$

$$\left(\alpha_1 u + \alpha_2 \frac{\partial u}{\partial n}\right)_S = \varphi(M) \tag{11}$$

where

$$\alpha_1 = \alpha_1(M), \quad \alpha_2 = \alpha_2(M), \quad \alpha_1, \alpha_2 \geqslant 0 \quad \text{and} \quad \alpha_1^2+\alpha_2^2 \neq 0$$

One of these methods is the method of separation of variables. We shall illustrate it by the following two examples.

*Example 1* Find the function $u(r, \varphi)$ which is harmonic within the circle $D_R$ of radius $R$, continuous in the closed domain $\bar{D}_R$ and assumes on the boundary of this region ($r = R$) the prescribed values $f(\varphi)$, i.e.

$$\nabla^2 u = 0 \tag{12}$$

$$u(R, \varphi) = f(\varphi) \tag{13}$$

Since the required solution $u(r, \varphi)$ is unique, it must be periodic in $\varphi$ with a period of $2\pi$, i.e.

$$u(r, \varphi+2\pi) \equiv u(r, \varphi) \tag{14}$$

Since the solution is continuous in the closed domain $\bar{D}_R$, it follows that it must be bounded in $\bar{D}_R$.

We shall seek the solutions of Equation (12) in the form $\Phi(r)$ $\Psi(\varphi)$ which are bounded in $\bar{D}_R$ and are periodic in $\varphi$ (with a period of $2\pi$). Let us write down the Laplacian in polar coordinates

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{1}{r^2}u_{\varphi\varphi} = 0 \tag{12a}$$

and separate the variables so that

$$r\frac{\mathrm{d}}{\mathrm{d}r}(r\Phi')-\lambda\Phi = 0 \tag{15}$$

$$\Psi''+\lambda\Psi = 0 \tag{16}$$

Using (14) we find that

$$\Psi(\varphi+2\pi) \equiv \Psi(\varphi) \tag{17}$$

For $\lambda < 0$, Equation (16) does not have solutions satisfying (17). Consequently, $\lambda \geqslant 0$. For $\lambda > 0$ we find that

$$\Psi(\varphi) = A\sin\sqrt{\lambda}\varphi+B\cos\sqrt{\lambda}\varphi$$

From (17) we find that $\sqrt{\lambda}\,2\pi = 2\pi n$ and hence $\lambda_n = n^2$, where $n$ is an arbitrary non-negative integer. Therefore, the eigenvalues of the problem (16)–(17) are

$$\lambda_n = n^2 \qquad (n = 0, 1, 2, \ldots)$$

and the corresponding eigenfunctions are

$$1, \sin\varphi, \cos\varphi, \ldots, \sin n\varphi, \cos n\varphi, \ldots$$

For $\lambda = 0$ the general solution of (16) is

$$\Psi(\varphi) = A_0\varphi + B_0$$

It is only for $A_0 = 0$ that this will satisfy the condition given by (17). Therefore, $\lambda = 0$ corresponds to the eigenfunction $\Psi_0(\varphi) \equiv 1$.

Consider now Equation (15). For $\lambda = n^2$ we have

$$r^2\Phi'' + r\Phi' - n^2\Phi = 0$$

The general solution of this equation is of the form

$$\Phi_n(r) = C_n r^n + \frac{D_n}{r^n} \qquad (n > 0) \tag{18}$$

$$\Phi_0(r) = C_0 + D_0 \ln\frac{1}{r} \qquad (n = 0) \tag{19}$$

Since the required solution is bounded, we must set $D_n = 0$ ($n = 0, 1, \ldots$) in Equations (18) and (19).

Therefore, the bounded solutions of ($12_1$) which are of the form $\Phi(r)\,\Psi(\varphi)$ and satisfy the condition given by (14) will be the functions

$$u_n = r^n(A_n \cos n\varphi + B_n \sin n\varphi)$$

The solution of the problem (12)–(14) can now be written in the form of the series

$$u(r, \varphi) = \sum_{n=0}^{\infty} r^n(A_n \cos n\varphi + B_n \sin n\varphi) \tag{20}$$

The coefficients $A_n$ and $B_n$ can be found from (13) using the orthogonality of the eigenfunctions within the range $[0, 2\pi]$ with the weight $\rho \equiv 1$:

$$f(\varphi) = \sum_{n=0}^{\infty} (A_n \cos n\varphi + B_n \sin n\varphi) R^n \tag{21}$$

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\, d(\xi), \qquad A_n = \frac{1}{\pi}\int_0^{2\pi} f(\xi) \cos n\xi \, d\xi$$

$$B_n = \frac{1}{\pi}\int_0^{2\pi} f(\xi) \sin n\xi \, d\xi \qquad (n = 1, 2, \ldots) \tag{22}$$

*Remark 1* The series (20) with the coefficients given by (22) can readily be summed. However, we shall not do this here since the problem (12)–(14) will be solved by another method in Section 5.4. This method yields the result in a closed form.

*Remark 2* The solution of the boundary-value problem (12)–(14) outside the circle can be represented by the series

$$u(r, \varphi) = \sum_{n=0}^{\infty} \frac{1}{r^n} (A_n \cos n\varphi + B_n \sin n\varphi)$$

whose coefficients can be determined from the condition given by (13). For the annular region between two concentric circles $R_1$ and $R_2$, the solution can be represented by the series

$$u = \sum_{n=1}^{\infty} \left( C_n r^n + \frac{D_n}{r^n} \right) (A_n \cos n\varphi + B_n \sin n\varphi) + A_0 + B_0 \ln r$$

whose coefficients ($A_0$, $B_0$, $C_n A_n$, $C_n B_n$, $D_n A_n$ and $D_n B_n$) can be determined from the boundary conditions

$$u(R_1, \varphi) = f_1(\varphi), \qquad u(R_2, \varphi) = f_2(\varphi)$$

The reader is recommended to write out the corresponding formulae for these coefficients.

*Example 2* Solve the boundary-value problem

$$\nabla^2 u = 0 \tag{23}$$

$$u(0, y) = u(l, y) = 0 \tag{24}$$

$$u(x, 0) = f_1(x), \qquad u(x, b) = f_2(x) \tag{25}$$

in the rectangular domain $\{0 \leqslant x \leqslant l, \ \ 0 \leqslant y \leqslant b\}$.

We shall seek the solutions of Equation (23) in the form $\Phi(x)$ $\Psi(y)$ which satisfy only the homogeneous boundary conditions (24). Substituting this form into (23) and separating the variables, we obtain

$$\frac{\Phi''}{\Phi} + \frac{\Psi''}{\Psi} = 0$$

This will be an identity if $\Phi''/\Phi = -\lambda$, $\Psi''/\Psi = \lambda = \text{const}$.

We thus obtain the following equations for the two functions:

$$\Phi'' + \lambda\Phi = 0 \tag{26}$$

$$\Psi'' - \lambda\Psi = 0 \tag{27}$$

From (24) we find that

$$\Phi(0) = \Phi(l) = 0 \tag{28}$$

The problem defined by (26), (28) has only positive eigenvalues:

$$\lambda_n = \frac{\pi^2 n^2}{l^2} \qquad (n = 1, 2, \ldots)$$

The corresponding eigenfunctions are $\Phi_n(x) = \sin(\pi n/l)\, x$.

Consider now Equation (27). For $\lambda = \lambda_n$ it has the general solution

$$\Psi_n(y) = C_n \cosh \sqrt{\lambda_n} y + D_n \sinh \sqrt{\lambda_n} y$$

Consequently, the solutions of (23) which satisfy the boundary condition given by (24) are of the form

$$u_n(x, y) = \Phi_n(x) \Psi_n(y)$$

The solution of the problem defined by (23)–(25) can be represented by the series

$$u(x, y) = \sum_{n=1}^{\infty} (C_n \cosh \sqrt{\lambda_n} y + D_n \sinh \sqrt{\lambda_n} y) \sin \frac{\pi n}{l} x$$

The coefficients of this series can be determined from the boundary conditions (25):

$$f_1(x) = \sum_{n=1}^{\infty} C_n \sin \frac{\pi n}{l} x$$

and

$$f_2(x) = \sum_{n=1}^{\infty} \left( C_n \cosh \frac{\pi n}{l} b + D_n \sinh \frac{\pi n}{l} b \right) \sin \frac{\pi n}{l} x$$

Hence,

$$C_n = \frac{2}{l} \int_0^l f_1(\xi) \sin \frac{\pi n}{l} \xi \, d\xi$$

and

$$C_n \cosh \frac{\pi n}{l} b + D_n \sinh \frac{\pi n}{l} b = \frac{2}{l} \int_0^l f_2(\xi) \sin \frac{\pi n}{l} \xi \, d\xi$$

In Sections 11.1 and 12.4 we shall quote other examples of the application of the method of separation of variables for equations of the elliptic type, which will require the use of special functions.

Another method of solving the boundary-value problems for elliptic equations is the method of Green's functions. It consists of the following. To begin with, one finds the solution of the problem (10)–(11) for special values of the functions $f(M)$ and $\varphi(M)$. In particular, one solves the problem

$$L(G) = -\delta(M, P) \tag{29}$$

$$\left(\alpha_1 G + \alpha_2 \frac{\partial G}{\partial n}\right)_S = 0 \tag{30}$$

This solution is called the Green's function for the problem (10)–(11).

We shall require that the required function $G(M, P)$ be continuous (together with its first-order partial derivatives if $\alpha_2 \neq 0$) everywhere in a closed domain $\bar{D}$ except, perhaps, for the point $P$ at which $G$ may have a singularity.

If Green's function has been found, it can be used to find the solution of the original problem (10)–(11). This can be done with the aid of Green's formula as follows

$$\int_D \{GL[u] - uL[G]\} \, d\tau = \int_S k\left(G \frac{\partial u}{\partial n} - u \frac{\partial G}{\partial n}\right) d\sigma \tag{31}$$

Here and in the ensuing analysis the derivative $\partial/\partial n$ is evaluated in the direction of the outward normal to $S$. Since in the domain $D$, $L[u] = f(M)$ and $L[G] = -\delta(M, P)$, the last expression can be written in the form

$$\int_D f(M) G(M, P) \, d\tau_M + \int_D u(M) \delta(M, P) \, d\tau_M = \int_S k\left(G \frac{\partial u}{\partial n} - u \frac{\partial G}{\partial n}\right) d\sigma_M$$

The second integral on the left-hand side is equal to $u(P)$ in view of the properties of the $\delta$ function. The last relationship can therefore be written in the form

$$u(P) = \int_S k\left(G \frac{\partial u}{\partial n} - u \frac{\partial G}{\partial n}\right) d\sigma_M - \int_D G(M, P) f(M) \, d\tau_M \tag{32}$$

where the integration is performed over the coordinates of the point $M$.

For the first boundary-value problem ($\alpha_1 \equiv 1, \alpha_2 \equiv 0$)

$$G|_S = 0, \qquad u|_S = \varphi$$

and from Equation (32) we obtain the solution of the problem (10)–(11):

$$u(P) = -\int_S k\varphi(M)\frac{\partial G}{\partial n}\,\mathrm{d}\sigma_M - \int_D G(M, P)f(M)\,\mathrm{d}\tau_M \tag{33}$$

For the second boundary problem ($\alpha_1 \equiv 0, \alpha_2 \equiv 1$)

$$\left.\frac{\partial G}{\partial n}\right|_S = 0, \qquad \left.\frac{\partial u}{\partial n}\right|_S = \varphi(M)$$

and from (32) we obtain the solution of the problem (10)–(11):

$$u(P) = \int_S k\varphi(M)G(M, P)\,\mathrm{d}\sigma_M - \int_D G(M, P)f(M)\,\mathrm{d}\tau_M \tag{34}$$

For the third boundary-value problem ($\alpha_1 \neq 0, \alpha_2 \neq 0$) we have

$$\left.\frac{\partial G}{\partial n}\right|_S = -\left.\frac{\alpha_1}{\alpha_2}G\right|_S, \qquad \left.\frac{\partial u}{\partial n}\right|_S = \left.\frac{-\alpha_1}{\alpha_2}u\right|_S + \left.\frac{\varphi(M)}{\alpha_2}\right|_S$$

and, in this case, (32) yields

$$u(P) = \int_S \frac{k\varphi(M)}{\alpha_2(M)}G(M, P)\,\mathrm{d}\sigma_M - \int_D G(M, P)f(M)\,\mathrm{d}\tau_M \tag{35}$$

Therefore, the original boundary-value problem (10)–(11) has been reduced to the determination of Green's function. Methods of finding these functions will be discussed later.

We shall now consider some of the properties of Green's functions.

Green's functions are symmetric, i.e.

$$G(M, P) = G(P, M)$$

To prove this let us use Green's formula for $G_1 = G(M, P_1)$ and $G_2 = G(M, P_2)$ where $P_1$ and $P_2$ are arbitrary fixed points in the domain $D$. We have

$$\int_D \{G_1 L[G_2] - G_2 L[G_1]\}\,\mathrm{d}\tau_M = \int_S k\left(G_1\frac{\partial G_2}{\partial n} - G_2\frac{\partial G_1}{\partial n}\right)\mathrm{d}\sigma_M$$

The left-hand side is equal to

$$-\int_D \{G(M, P_1)\delta(M, P_2) - G(M, P_2)\delta(M, P_1)\}\,d\tau_M$$

$$= G(P_1, P_2) - G(P_2, P_1)$$

and, consequently,

$$G(P_1, P_2) - G(P_2, P_1) = \int_S k\left(G_1\frac{\partial G_2}{\partial n} - G_2\frac{\partial G_1}{\partial n}\right) d\sigma_M$$

The integral on the right-hand side is zero; in fact, if we are dealing with the first (or second) boundary-value problem, this follows from the boundary conditions for $G_1$ and $G_2$ ($G|_S = 0$ or $\partial G/\partial n|_S = 0$). If we are dealing with the third boundary-value problem, then by expressing $\partial G_1/\partial n$ and $\partial G_2/\partial n$ in terms of $G_1$ and $G_2$, and substituting these values into the integrand, we obtain

$$\left(G_1\frac{\partial G_2}{\partial n} - G_2\frac{\partial G_1}{\partial n}\right)_S = -\frac{\alpha_1}{\alpha_2}G_1G_2 + \frac{\alpha_1}{\alpha_2}G_2G_1 \equiv 0$$

Therefore, $G(P_1, P_2) = G(P_2, P_1)$.

We must now investigate the properties of Green's function at the point $P$. We shall confine our attention to the case when $L[u] \equiv \nabla^2$. In this case, Green's function has a singularity of the form $1/4\pi r_{MP}$ in three-dimensional space or $(1/2\pi)[\ln(1/r_{MP})]$ on a plane.

In view of the structure of the equation $\nabla^2 G = -\delta(M, P)$, which is satisfied by Green's function, one would expect that this function may be written in the form

$$G(M, P) = \psi(r_{MP}) + v(M, P)$$

where $v$ is harmonic in $D$ (as a function of $M$), and $\psi(r_{MP})$ has a singularity at the point $P$, i.e. at $r_{MP} = 0$, and should satisfy the equation $\nabla^2\varphi = -\delta(M, P)$.

To be specific, consider the three-dimensional case. Let $D_P^R$ represent the spherical region bounded by the surface $S_P^R$ and centred on the point $P$. Let the radius of the region be $R$. If we integrate the identity

$$\nabla^2\psi \equiv -\delta(M, P)$$

over the domain $D_P^R$ ($D_P^R \subset D$), we obtain

$$\int\limits_{D_P^R} \nabla^2\psi \, \mathrm{d}\tau_M = -1$$

By Green's formula the integral on the left-hand side is equal to

$$\int\limits_{S_P^R} \frac{\partial\psi}{\partial n} \, \mathrm{d}\sigma_M = \int\limits_{S_P^R} \frac{\mathrm{d}\psi}{\partial r} \, \mathrm{d}\sigma_M$$

and, therefore,

$$\int\limits_{S_P^R} \frac{\mathrm{d}\psi}{\mathrm{d}r} \, \mathrm{d}\sigma_M = -1$$

On the sphere $S_P^R$ the function $\mathrm{d}\psi/\mathrm{d}r$ has a constant value and, therefore,

$$\left.\frac{\mathrm{d}\psi}{\mathrm{d}r}\right|_{r=R} \int\limits_{S_P^R} \mathrm{d}\sigma = -1 \qquad \text{for} \quad 4\pi R^2 \frac{\mathrm{d}\psi(R)}{\mathrm{d}R} = -1$$

Hence,

$$\psi(R) = \frac{1}{4\pi R}$$

It follows that Green's function $G(M, P)$ is given by

$$G(M, P) = \frac{1}{4\pi r_{MP}} + v(M, P) \tag{36}$$

and, consequently, it has a singularity of the form $1/4\pi r_{MP}$ at the point $P$.

On the plane, the function $G(M, P)$ is of the form

$$G(M, P) = \frac{1}{2\pi} \ln\left[\frac{1}{r_{MP}} + v(M, P)\right] \tag{$36_1$}$$

The function $v(M, P)$ is defined as the solution of the problem

$$\nabla^2 v = 0, \left(\alpha_1 v + \alpha_2 \frac{\partial v}{\partial n}\right)_S = -\left[\frac{\alpha_1}{r_{MP}} + \alpha_2 \frac{\partial\left(\frac{1}{r_{MP}}\right)}{\partial n}\right]_S \frac{1}{4\pi}$$

It is unique for the first (and the third) boundary-value problem and is determined to within an additive constant for the second boundary-value problem.

Green's function is determined in a similar way for external boundary-value problems. It is again symmetric and has the same singularities for $L[u] = \nabla^2 u + qu$.

Using Equation (36) one can readily provide a physical interpretation for Green's function in the case of the operator $\nabla^2 u$. We shall do this for the first boundary-value problem.

Suppose that the surface $S$ bounding the domain $D$ is in the form of an earthed conductor. Let us place an electric charge of magnitude $1/4\pi$ at the point $P$ inside $D$. This charge will induce a certain charge distribution on $S$ and the potential in $D$ will be equal to the sum of the potential $1/4\pi r_{MP}$ due to the point charge and the potential $v(M, P)$ due to the induced charge. This sum is, in fact, equal to $G(M, P)$.

It follows that $G(M, P)$ can be interpreted as the potential due to a point charge placed inside an earthed closed conducting surface. In this interpretation the symmetry of Green's function is an expression of the reciprocity theorem for the point at which the charge s located and the point of observation.

*Remark 3* Green's function determined in this way does not always exist. For example, Green's function for the second internal boundary-value problem for the Laplace operator $L[u] \equiv \nabla^2 u$ does not exist since one cannot find a function $v(M)$ $(G = 1/4\pi r + v)$ which is harmonic in $D$, continuous in $\bar{D}$ together with its first-order partial derivatives and satisfies the condition

$$\left.\frac{\partial v}{\partial n}\right|_S = \varphi(M) = -\frac{1}{4\pi}\frac{\partial}{\partial n}\left(\frac{1}{r_{MP}}\right).$$

This is because the necessary condition $\int\limits_S \varphi \, \mathrm{d}\sigma = 0$ is not satisfied. In this case, Green's function can be determined as the solution of the boundary-value problem

$$\nabla^2 G = -\delta(M, P), \qquad \left.\frac{\partial G}{\partial n}\right|_S = \frac{1}{S_0}$$

where $S_0$ is the area of the surface $S$. This function exists and is defined to within an additive constant. Using (32), we can find the solution $u(P)$ of the second boundary-value problem (10)–(11)

$$u(P) = \int\limits_S k(M) G(M, P) \varphi(M) \, \mathrm{d}\sigma_M - \int\limits_D G(M, P) f(M) \, \mathrm{d}\tau_M - \int\limits_S \frac{ku}{S_0} \, \mathrm{d}\sigma$$

or

$$u(P)=\int_S k(M)G(M,P)\varphi(M)\,\mathrm{d}\sigma_M-\int_D G(M,P)f(M)\,\mathrm{d}\tau_M+C$$

where

$$C=\text{const}\qquad\qquad \left[C=-\int_S k(M)\frac{u(M)}{S_0}\,\mathrm{d}\sigma_M\right]$$

## 6.4 DERIVATION OF GREEN'S FUNCTIONS. POISSON'S INTEGRAL

One of the methods of constructing Green's functions is the method of images. We shall illustrate it by the following examples.

*Example 1* Derive Green's function for the first boundary-value problem for a half-space bounded by a plane $Q$ (we can assume without loss of generality that this plane lies in the $z = 0$ plane).

Let $P$ be the singular point of Green's function. Since

$$G(M,P)=\frac{1}{4\pi r_{MP}}+v$$

the problem reduces to the determination of a function $v$ which is harmonic in the half-space under consideration (for example, $z > 0$) and is equal to $-1/4\pi r_{MP}$ on its boundary. It is clear that this function is $v = -1/4\pi r_{MP_1}$, where $P_1$ and $P$ are symmetric with respect to the plane $Q$. In fact, the function $-1/4\pi r_{MP_1}$ is harmonic in the half-space $z > 0$ and is equal to $1/4\pi r_{MP}$ at the points $M \in Q$ since for such points $r_{MP} = r_{MP_1}$. Therefore, the required Green's function is

$$G(M,P)=\frac{1}{4\pi r_{MP}}-\frac{1}{4\pi r_{MP_1}}$$

This method of deriving Green's function for a half-space bounded by a plane is suggested by the physical interpretation of Green's function given in Section 6.3. In point of fact, if we place point charges $1/4\pi$ and $-1/4\pi$ at the symmetric points $P$ and $P_1$, the potential produced by these will then be a function which is harmonic everywhere except for the points $P$ and $P_1$, and is zero on the plane $Q$.

Similarly, for a half-plane bounded by a straight line $l$, Green's function is of the form

$$G(M, P) = \frac{1}{2\pi} \ln \frac{1}{r_{MP}} - \frac{1}{2\pi} \ln \frac{1}{r_{MP_1}}$$

where the points $P_1$ and $P$ are symmetric with respect to the straight line $l$.

*Example 2* Derive Green's functions for the first boundary-value problem in the case of two straight rays $l_1$ and $l_2$ forming a right-angle.

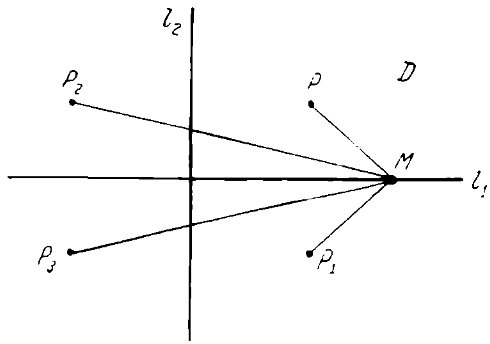



*Fig. 6.2*

Let $P$ be a singular point of Green's function. The images of $P$ in $l_1$ and $l_2$ will be the points $P_1$ and $P_2$ (Fig. 6.2). Let $P_3$ be the image of $P_1$ and $P_2$ with respect to the continuations of the two lines $l_1$ and $l_2$. Green's function will then be given by

$$G(M, P) = \frac{1}{2\pi} \ln \frac{1}{r_{MP}} - \frac{1}{2\pi} \ln \frac{1}{r_{MP_1}} - \frac{1}{2\pi} \ln \frac{1}{r_{MP_2}} + \frac{1}{2\pi} \ln \frac{1}{r_{MP_3}}$$

In fact, in this case, the function $v$ is equal to

$$-\frac{1}{2\pi} \ln \frac{1}{r_{MP_1}} - \frac{1}{2\pi} \ln \frac{1}{r_{MP_2}} + \frac{1}{2\pi} \ln \frac{1}{r_{MP_3}}$$

It is harmonic within the right-angle $D$ (as a function of the point $M$) and is equal to $-(1/2\pi)[\ln(1/r_{MP})]$ on its sides. This follows from the fact that if $M \in l_1$ then $r_{MP} = r_{MP_1}$, $r_{MP_2} = r_{MP_3}$, if $M \in l_2$, then $r_{MP} = r_{MP_2}$, $r_{MP_1} = r_{MP_3}$.

*Example 3* Determine Green's function for the first (internal) boundary-value problem for the circle

$$G(M, P) = \frac{1}{2\pi} \ln \frac{1}{r_{MP}} + v$$

The problem reduces to the determination of the function $v$, which is harmonic within the circle and is equal to $-(1/2\pi)[\ln(1/r_{MP})]$ on its boundary.

Let $P$ be the singular point of Green's function, and let $P_1$ be the image of $P$ with respect to the boundary of the region (the circle $C$). $P_1$ is the image of $P$ with respect to the circle $C$ if both these points lie on the same ray drawn through the centre of the circle

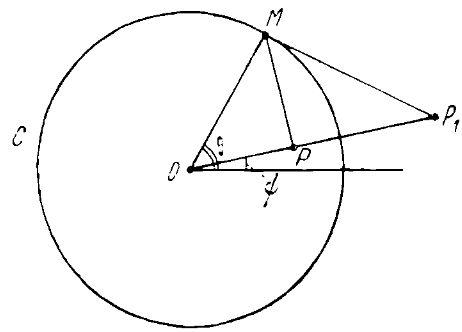



*Fig. 6.3*

and the product of their distances, $\rho_1$ and $\rho$, is equal to the square of the radius, i.e. $\rho\rho_1 = R^2$. If the point $M$ lies on $C$, then it is evident from Fig. 6.3 that

$$r_{MP_1} = \frac{R}{\rho} r_{MP} \tag{37}$$

since the triangles $OMP_1$ and $OMP$ are similar. Therefore, the function

$$v = -\frac{1}{2\pi} \ln \frac{R}{\rho r_{MP_1}}$$

is the required function. Consequently, Green's function for the first (internal) boundary-value problem for a circle is of the form

$$G(M, P) = \frac{1}{2\pi} \ln \frac{1}{r_{MP}} - \frac{1}{2\pi} \ln \frac{R}{\rho r_{MP_1}} \tag{38}$$

*Example 4* Solve the first internal boundary-value problem for Laplace's equation $\nabla^2 u = 0$ in a circle.

The required solution is given by

$$u(P) = -\int_C \varphi(S) \frac{\partial G}{\partial n} \, dS \tag{39}$$

which is obtained from (33) with $f(M) \equiv 0$. In the case under consideration, Green's function $G$ is given by (38). Let us evaluate $\partial G/\partial n$:

$$\frac{\partial G}{\partial n} = \frac{\partial G}{\partial r}\cos(\boldsymbol{n}, \boldsymbol{r}) = \frac{1}{2\pi}\frac{1}{r_{MP}}\cos(\boldsymbol{n}, \boldsymbol{r}_{MP}) - \frac{1}{2\pi}\frac{1}{r_{MP_1}}\cos(\boldsymbol{n}, \boldsymbol{r}_{MP_1})$$

From the triangles $OMP$ and $OMP_1$ (Fig. 6.3), we find that

$$\cos(\boldsymbol{n}, \boldsymbol{r}_{MP}) = \frac{R^2+r_{MP}^2-\rho^2}{2Rr_{MP}}, \qquad \cos(\boldsymbol{n}, \boldsymbol{r}_{MP_1}) = \frac{R^2+r_{MP_1}^2-\rho_1^2}{2Rr_{MP_1}}$$

and, therefore,

$$\left.\frac{\partial G}{\partial n}\right|_C = \frac{1}{2\pi}\left(\frac{R^2+r_{MP}^2-\rho^2}{2Rr_{MP}^2} - \frac{R^2+r_{MP_1}^2-\rho_1^2}{2Rr_{MP_1}^2}\right)$$

Substituting for $r_{MP_1}$ from (37) and for $\rho_1$ from the formula $\rho_1 = R^2/\rho$, we obtain

$$\left.\frac{\partial G}{\partial n}\right|_C = \frac{1}{2\pi R}\cdot\frac{R^2-\rho^2}{r_{MP}^2}$$

From the triangle $OPM$ we find that $r_{MP}^2 = R^2+\rho^2-2R\rho\cos(\theta-\psi)$ and therefore

$$\left.\frac{\partial G}{\partial n}\right|_C = \frac{1}{2\pi R}\,\frac{R^2-\rho^2}{R^2+\rho^2-2R\rho\cos(\theta-\psi)}$$

Substituting this value into (39) we obtain the Poisson integral

$$u(P) = \frac{1}{2\pi}\int_0^{2\pi}\frac{(R^2-\rho^2)\varphi(\theta)\,d\theta}{R^2+\rho^2-2R\rho\cos(\theta-\psi)} \tag{40}$$

where $(\rho, \psi)$ are the polar coordinates of $P$, and $(R, \theta)$ are the polar coordinates of the point $M$ on $C$.

## PROBLEMS

1. Derive Green's function for the first external boundary-value problem: (a) for a circle and (b) for a sphere ($L[u] \equiv \nabla^2 u$).

2. Derive Green's function for the first internal boundary-value problem for a circular sector $0 \leqslant \varphi \leqslant \pi/n$ ($L[u] \equiv \nabla^2 u$).

3. Derive Green's function for the first internal boundary-value problem for (a) a spherical layer $R_1 \leqslant r \leqslant R_2$ and (b) for a ring $R_1 \leqslant r \leqslant R_2$.

4. Derive Green's function for the first internal boundary-value problem for a plane layer $0 \leqslant z \leqslant h$ $(L[u] \equiv \nabla^2 u)$.

5. Using the principle of the maximum and minimum of harmonic functions, show that Green's function for the first boundary-value problem for the domain $D$ is positive in $D$ $(L[u] \equiv \nabla^2 u)$.

6. Using the Poisson integral prove the following theorems.

(a) Any harmonic function which is positive on a plane is equal to a constant.

(b) Harnak's theorem. Let $\{u_i(x, y)\}$, $i = 1, 2, \ldots$, be harmonic functions in a finite domain $D$ bounded by the contour $\Gamma$, which are continuous in $\bar{D}$. If the series $\sum_{i=1}^{\infty} u_i(x, y)$ is uniformly convergent on $\Gamma$, it will converge uniformly in $D$ and its sum will be a harmonic function in $D$.

# 7

# Potentials

The method of separation of variables and the method of Green's functions was discussed in Chapter 6 in connection with elliptic equations. The third method for solving boundary-value problems is the method of integral equations, in which the solution is sought in the form of certain special integrals (potentials) with unknown charge (mass) density distributions. In the present chapter, we shall consider the simplest properties of potentials and their application to the solution of boundary-value problems.

## 7.1 VOLUME POTENTIAL

**7.1.1** The electrostatic potential (in space) due to a charge $e$ at a point $P$ is given by (at an arbitrary point $M$)

$$u(M) = \frac{e}{r_{MP}}$$

where $r_{MP}$ is the distance between $M$ and $P$.

If there are charges $e_1, e_2, \ldots, e_n$ at the points $P_1, P_2, \ldots, P_n$, the electrostatic potential due to these charges is given by

$$u(M) = \frac{e_1}{r_{MP_1}} + \frac{e_2}{r_{MP_2}} + \ldots + \frac{e_n}{r_{MP_n}} \tag{1}$$

Suppose that charges are distributed with a density $\rho(P)$ in a domain $D$. A small volume element $d\tau_p$ containing the point $P$ will

contain the amount of charge $\rho(P)\,\mathrm{d}\tau_P$. The potential due to this charge is approximately equal to

$$\frac{\rho(P)}{r_{MP}}\,\mathrm{d}\tau_P$$

The potential due to all the charges in $D$ is given by

$$u(M) = \int_D \frac{\rho(P)}{r_{MP}}\,\mathrm{d}\tau_P \tag{2}$$

This integral is called the volume potential. In two-dimensional space (plane) the volume potential assumes the form

$$u(M) = \int_D \rho(P)\ln\left(\frac{1}{r_{MP}}\right)\mathrm{d}s_P \tag{3}$$

**7.1.2** The volume potential is thus an improper integral. Consider the more general improper integral

$$u(M) = \int_D f(M, P)\,\mathrm{d}\tau_P \tag{4}$$

where $f(M, P)$ is a continuous function of the two points $M$ and $P$, $M \neq P$, which becomes infinite as $M \to P$. We shall call the integral (4) uniformly convergent in the neighbourhood of the point $M_0$ if for any $\varepsilon > 0$ there exists a number $\delta$ such that:

1. for any domain $D^\delta_{M_0}$ containing $M_0$ and having a diameter $d$ less than $\delta$, i.e. $\mathrm{d}(D^\delta_{M_0}) < \varepsilon$, and

2. for all points $M$ separated from $M_0$ by a distance less than $\delta(\overline{MM_0} < \delta)$,

$$\left|\int_{D^\delta_{M_0}} f(M, P)\,\mathrm{d}\tau_P\right| < \varepsilon$$

This idea lies at the basis of proofs of a number of properties of potentials. The main property of a uniformly convergent improper integral is expressed by the following theorem.

*Theorem* An improper integral which is uniformly convergent in the neighbourhood of a point $M_0$ is continuous at that point.

*Proof* Consider the difference

$$u(M)-u(M_0) = u_1(M)-u_1(M_0)+\{u_2(M)-u_2(M_0)\}$$

where

$$u_1(M) = \int_{D^\delta_{M_0}} f(M, P)\, \mathrm{d}\tau_P, \qquad u_2(M) = \int_{D-D^\delta_{M_0}} f(M, P)\, \mathrm{d}\tau_P$$

Since the integral (4) converges uniformly in the neighbourhood of $M_0$, it follows that for arbitrary $\varepsilon > 0$ it is possible to find a number $\delta$ such that for the domain $D^\delta_{M_0}$ with $\mathrm{d}(D^\delta_{M_0}) < \delta$ and for all points $M$, separated from $M_0$ by a distance less than $\delta$, we have the following inequalities

$$|u_1(M)| < \frac{\varepsilon}{3}, \qquad |u_1(M_0)| < \frac{\varepsilon}{3} \tag{5}$$

Since $M_0 \notin D - D^\delta_{M_0}$, the integral $u_2(M)$ is not an improper integral and the function $u_2(M)$ is continuous at $M_0$. Consequently, for the same $\varepsilon$ we can find a number $\delta_1$ such that for all the points $M$, separated from $M_0$ by distances less than $\delta_1$, we have

$$|u_2(M) - u_2(M_0)| < \frac{\varepsilon}{3} \tag{6}$$

Let $\delta_2 = \min\{\delta, \delta_1\}$. For all points $M$, such that $\overline{MM_0} < \delta_2$, we then have inequalities (5) and (6) and, consequently, the inequality

$$|u(M) - u(M_0)| < \varepsilon$$

This proves the theorem.

We note that the uniform convergence of an improper integral ensures that it is convergent at the point $M_0$.

**7.1.3** Let us now consider the simplest properties of the volume potential with density $\rho(P)$, $|\rho(P)| \leqslant A$.

*Property 1* The volume potential exists and is continuous everywhere.

If the point $M_0$ does not belong to $D$, the integral $u(M_0)$ is not an improper integral. Since the integrand, regarded as a function of $M$, is continuous at $M_0$, the integral $u(M)$ must be continuous at this point. If $M_0 \in D$, then according to the theorem of Section 7.2 and the remark at the end of Section 7.2 it is sufficient to know that

the integral converges uniformly in the neighbourhood of $M_0$. Consider the integral

$$\int\limits_{D^\delta_{M_0}} \frac{\rho(P)}{r_{MP}}\,\mathrm{d}\tau_P$$

It is evident that

$$\left|\int\limits_{D^\delta_{M_0}} \frac{\rho(P)}{r_{MP}}\,\mathrm{d}\tau_P\right| \leqslant \int\limits_{D^\delta_{M_0}} \frac{|\rho|}{r_{MP}}\,\mathrm{d}\tau_P \leqslant A \int\limits_{D^\delta_{M_0}} \frac{\mathrm{d}\tau_P}{r_{MP}} < A \int\limits_{T^{2\delta}_M} \frac{\mathrm{d}\tau_P}{r_{MP}}$$

where $T^{2\delta}_M$ is a spherical region centred on $M$ and of radius $2\delta$ $(D^\delta_{M_0} \subset T^{2\delta}_M)$. Transforming to polar coordinates, we have

$$A \int\limits_{T^{2\delta}_M} \frac{\mathrm{d}\tau_P}{r_{MP}} = A \int\limits_0^{2\delta}\int\limits_0^{\pi}\int\limits_0^{2\pi} r \sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi = 8A\pi\delta^2 \qquad (r = r_{MP})$$

Therefore $\left|\int\limits_{D^\delta_{M_0}} \frac{\rho(P)}{r_{MP}}\,\mathrm{d}\tau_P\right| < 8A\pi\delta^2$. To ensure that this integral is less than a given number $\varepsilon$, it is sufficient to take $\delta < \sqrt{\varepsilon/8\pi A}$.

*Property 2* The first-order partial derivatives of the volume potential with respect to the coordinates of the point $M$ are everywhere continuous.

If $M_0 \notin D$, the integral $u(M_0)$ is not an improper integral. Since the integrand, regarded as a function of the point $M$, has continuous first-order partial derivatives with respect to the coordinates of $M$ at the point $M_0$, the integral $u(M)$ will also have this property and the derivatives can be evaluated by differentiating under the integral sign:

$$\frac{\partial u}{\partial x} = \int\limits_D \frac{(\xi - x)}{r^3_{MP}}\rho(P)\,\mathrm{d}\tau_P, \qquad \frac{\partial u}{\partial y} = \int\limits_D \frac{(\eta - y)}{r^3_{MP}}\rho(P)\,\mathrm{d}\tau_P$$

$$\frac{\partial u}{\partial z} = \int\limits_D \frac{(\zeta - z)}{r^3_{MP}}\rho(P)\,\mathrm{d}\tau_P \tag{7}$$

where $(\xi, \eta, \zeta)$ are the coordinates of $P$.

If $M_0 \in D$, then it is sufficient to show that the integrals of the derivatives on the right-hand sides of (7) converge uniformly in the neighbourhood of $M_0$. One can then differentiate under the integral

sign and the formulae given by (7) are valid for $\partial u/\partial x$, $\partial u/\partial y$ and $\partial u/\partial z$. To be specific, consider the integral

$$\int_D \frac{(\xi-x)\rho(P)}{r_{MP}^3}\,\mathrm{d}\tau_P$$

It is evident that

$$\left|\int_{D_{M_0}^\delta} \frac{(\xi-x)}{r_{MP}^3}\rho(P)\,\mathrm{d}\tau_P\right| \leqslant A\int_{D_{M_0}^\delta} \frac{|\xi-x|}{r_{MP}}\frac{\mathrm{d}\tau_P}{r_{MP}^2} \leqslant A\int_{D_{M_0}^\delta}\frac{\mathrm{d}\tau_P}{r_{MP}^2}$$

since $|\xi-x/r_{MP}| = |\cos(\widehat{r,n})| \leqslant 1$. Next

$$A\int_{D_{M_0}^\delta}\frac{\mathrm{d}\tau_P}{r_{MP}^2} \leqslant A\int_{T_M^{2\delta}}\frac{\mathrm{d}\tau_P}{r_{MP}^2} = A\int_0^{2\delta}\int_0^{\pi}\int_0^{2\pi}\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi = 8\pi A\delta$$

To satisfy the inequality

$$\left|\int_{D_{M_0}^\delta}\frac{(\xi-x)\rho(P)}{r_{MP}^3}\,\mathrm{d}\tau_P\right| < \varepsilon$$

it is sufficient to take $\delta < \varepsilon/8\pi A$.

*Property 3* The volume potential is a harmonic function outside the domain $D$ in which the charges (masses) are located.

This property follows from the fact that for points $M \notin D$ the integral (2) is not an improper integral and, therefore, the Laplace operator can be taken out from under the integral sign:

$$\nabla^2 u = \left(\int_D \frac{\rho(P)}{r_{MP}}\,\mathrm{d}\tau_P\right) = \int_D \rho(P)\,\nabla^2\left(\frac{1}{r_{MP}}\right)\mathrm{d}\tau_P \equiv 0$$

since for points $M \notin D$ we have $\nabla^2(1/r_{MP}) \equiv 0$.

*Property 4* At points in the domain $D$ the volume potential satisfies the equation

$$\nabla^2 u = -4\pi\rho(M) \tag{8}$$

We shall suppose that $\rho(P)$ has bounded and integrable first-order partial derivatives $\partial\rho/\partial\xi$, $\partial\rho/\partial\eta$, $\partial\rho/\partial\zeta$.

*Proof* The second-order derivatives $\partial^2 u/\partial x^2$, $\partial^2 u/\partial y^2$, $\partial^2 u/\partial z^2$ cannot be obtained by differentiating the right-hand side of (7) under the integral sign since this yields divergent integrals, for example:

$$-\int_D \frac{\rho}{r_{MP}^3}\,\mathrm{d}\tau_P + \int \frac{3(\xi-x)^2\rho(P)}{r_{MP}^5}\,\mathrm{d}\tau_P$$

To establish the existence of the second-order derivatives we shall proceed as follows. Suppose that $M_0 \in D$; $T_{M_0}^\delta$ is a spherical region of radius $\delta$ centred on the point $M_0$ and bounded by the spherical surface $S_{M_0}^\delta$, where $T_{M_0}^\delta \subset D$. For points $M$ and $T_{M_0}^\delta$ we can write

$$u(M) = u_1(M) + u_2(M)$$

where

$$u_1(M) = \int_{T_{M_0}^\delta} \frac{\rho(P)}{r_{MP}}\,\mathrm{d}\tau_P, \qquad u_2(M) = \int_{D-T_{M_0}^\delta} \frac{\rho(P)}{r_{MP}}\,\mathrm{d}\tau_P$$

The integral $u_2(M)$ is not an improper integral.

In view of Property 3, it is harmonic at the point $M_0$, i.e. $\nabla^2 u_2|_{M=M_0} = 0$. Consequently, $\nabla^2 u|_{M=M_0} = u_1 \nabla^2|_{M=M_0}$ and therefore it is sufficient to consider the function $u_1(M)$.

The derivative

$$\frac{\partial u_1}{\partial x} = \int_{T_{M_0}^\delta} \frac{\xi - x}{r_{MP}^3}\,\mathrm{d}\tau_P = \int_{T_{M_0}^\delta} \rho(P)\frac{\partial}{\partial x}\left(\frac{1}{r_{MP}}\right)\mathrm{d}\tau_P$$

can also be written in the form

$$\frac{\partial u_1}{\partial x} = -\int_{T_{M_0}^\delta} \rho(P)\frac{\partial}{\partial \xi}\left(\frac{1}{r_{MP}}\right)\mathrm{d}\tau_P = \int_{T_{M_0}^\delta} \frac{\dfrac{\partial\rho}{\partial\xi}}{r_{MP}}\,\mathrm{d}\tau_P - \int_{T_{M_0}^\delta} \frac{\partial}{\partial\xi}\left(\frac{\rho}{r_{MP}}\right)\mathrm{d}\tau$$

Applying Green's formula to the second integral, we obtain

$$\frac{\partial u_1}{\partial x} = \int_{T_{M_0}^\delta} \frac{\dfrac{\partial\rho}{\partial\xi}}{r_{MP}}\,\mathrm{d}\tau_P - \int_{S_{M_0}^\delta} \frac{\rho(P)}{r_{MP}}\cos\alpha\,\mathrm{d}\sigma_P \tag{9}$$

where $\alpha$ is the angle between the direction of the outward normal to $S_{M_0}^\delta$ and the $x$ axis.

The first integral on the right of (9) is a volume potential with charge (mass) density $\rho_1(P) = \partial\rho/\partial\xi$. Therefore, in accordance

with Property 2 it has a continuous first-order derivative with respect to $x$. The second integral is not an improper integral and therefore possesses a continuous first-order derivative with respect to $x$ for any internal point $M$ of $T^{\delta}_{M_0}$. Consequently, $\partial u_1/\partial x$ has a continuous second-order derivative with respect to $x$ in $T^{\delta}_{M_0}$. At the same time,

$$\left.\frac{\partial^2 u_1}{\partial x^2}\right|_{M=M_0} = \int\limits_{T^{\delta}_{M_0}} \frac{(\xi-x_0)\dfrac{\partial\rho}{\partial\xi}}{r^3_{M_0P}}\, \mathrm{d}\tau_P - \int\limits_{S^{\delta}_{M_0}} \frac{\rho(P)}{r^3_{M_0P}}(\xi-x_0)\cos\alpha\, \mathrm{d}\sigma_P$$

However, $(\xi-x_0)/r_{M_0P} = \cos\alpha$, and therefore

$$\left.\frac{\partial^2 u_1}{\partial x^2}\right|_{M=M_0} = \int\limits_{T^{\delta}_{M_0}} \frac{(\xi-x_0)\dfrac{\partial\rho}{\partial\xi}}{r^3_{M_0P}}\, \mathrm{d}\tau_P - \int\limits_{S^{\delta}_{M_0}} \frac{\rho(P)}{r^2_{M_0P}}\cos^2\alpha\, \mathrm{d}\sigma_P \tag{10}$$

Similarly, we find that

$$\left.\frac{\partial^2 u_1}{\partial y^2}\right|_{M=M_0} = \int\limits_{T^{\delta}_{M_0}} \frac{(\eta-y_0)\dfrac{\partial\rho}{\partial\eta}}{r^3_{M_0P}}\, \mathrm{d}\tau_P - \int\limits_{S^{\delta}_{M_0}} \frac{\rho(P)}{r^2_{M_0P}}\cos^2\beta\, \mathrm{d}\sigma_P \tag{11}$$

$$\left.\frac{\partial^2 u_1}{\partial z^2}\right|_{M=M_0} = \int\limits_{T^{\delta}_{M_0}} \frac{(\zeta-z_0)\dfrac{\partial\rho}{\partial\zeta}}{r^3_{M_0P}}\, \mathrm{d}\tau_P - \int\limits_{S^{\delta}_{M_0}} \frac{\rho(P)}{r^2_{M_0P}}\cos^2\gamma\, \mathrm{d}\sigma_P \tag{12}$$

where $\beta$ and $\gamma$ are the angles between the normal to $S^{\delta}_{M_0}$ and the $y$ and $z$ axes, respectively.

Combining (10), (11) and (12), we have

$$\nabla^2 u|_{M=M_0} = \nabla^2 u_1|_{M=M_0} = \int\limits_{T^{\delta}_{M_0}} \frac{\dfrac{\partial\rho}{\partial\zeta}}{r^2_{M_0P}}\cos\alpha\, \mathrm{d}\tau_P + \int\limits_{T^{\delta}_{M_0}} \frac{\dfrac{\partial\rho}{\partial\eta}}{r^2_{M_0P}}\cos\beta\, \mathrm{d}\tau_P$$

$$+ \int\limits_{T^{\delta}_{M_0}} \frac{\dfrac{\partial\rho}{\partial\zeta}}{r^2_{M_0P}}\cos\gamma\, \mathrm{d}\tau_P - \int\limits_{S^{\delta}_{M_0}} \frac{\rho\, \mathrm{d}\sigma_P}{r^2_{M_0P}} \tag{13}$$

If we repeat the analysis leading to Property 2, we find that each of the integrals over the domain $T^{\delta}_{M_0}$ in (13) will not exceed $4\pi B\delta$, where $B$ is the upper bound of the functions $|\partial\rho/\partial\xi|$, $|\partial\rho/\partial\eta|$, $|\partial\rho/\partial\zeta|$, i.e.

$$\left| \int\limits_{T^{\delta}_{M_0}} \right| \leqslant 4\pi B\delta \tag{14}$$

Using the mean value theorem in (13) we obtain

$$\int\limits_{S^{\delta}_{M_0}} \frac{\rho}{r^2_{M_0P}} \,\mathrm{d}\sigma_P = 4\pi\rho(P^*) \tag{15}$$

where $P^* \in S^{\delta}_{M_0}$. Proceeding to the limit in (13) as $\delta \to 0$, and bearing in mind (14) and (15), we obtain

$$\nabla^2 u|_{M=M_0} = -4\pi\rho(M_0)$$

In the two-dimensional case the analogue of (8) is

$$\nabla^2 u = -2\pi\rho(M_0) \tag{16}$$

*Property 5* As the point of observation tends to infinity, the volume potential tends to zero (in the three-dimensional case, $D$ is a bounded domain).

To prove this property let us apply the mean value theorem to (2). We obtain

$$u(M) = \frac{1}{r_{MP^*}} \int\limits_D \rho \,\mathrm{d}\tau_P = \frac{m}{r_{MP^*}}$$

where $P^* \in D$, $m = \int \rho\,\mathrm{d}\tau_P$ is the total charge. This proves Property 5.

*Example* Let us find the volume potential due to a uniformly charged sphere $D$ of radius $R$. It is evident that the required potential is a function of the distance $R$ from the centre of the sphere to the point of observation:

$$u(M) = u(r)$$

Outside the sphere $D$ we have $\nabla^2 u = 0$ and, consequently, $u = C_1/r + C_2$. In view of Property 5 we have $u(r) \to 0 (r \to \infty)$. Consequently, $C_2 = 0$. Inside the sphere $D$ we have $\nabla^2 u = -4\pi\rho$ or $\mathrm{d}(r^2u')/\mathrm{d}r = -4\pi\rho r^2$. Consequently, $u(r) = (-2/3)\pi r^2\rho + (A/r) + B$ for $r \leqslant R$. Since the volume integral is bounded everywhere, it

follows that $A = 0$. From the condition that the potential and its first-order derivatives must be continuous, we find that

$$-\frac{2}{3}\pi R^2\rho + B = \frac{C_1}{R} \quad \text{and} \quad -\frac{4}{3}\pi R\rho = -\frac{C_1}{R^2}$$

and hence $C_1 = (4/3)\pi R^3\rho$, $B = 2\pi R^2\rho$. Therefore,

$$u(r) = \begin{cases} \dfrac{2}{3}\pi(3R^2 - r^2)\rho & r \leqslant R \\ \dfrac{4}{3}\pi\dfrac{R^3\rho}{r} & r \geqslant R \end{cases}$$

*Remark* The volume potential can be written in the form of the convolution (in the variables $x, y, z$) of the fundamental solution $1/4\pi r = (1/4\pi)(x^2+y^2+z^2)^{-1/2}$ of the Laplace equation $\nabla^2 u = 0$ $[\nabla^2(1/4\pi r) = -\delta(x, y, z)]$ and the function $4\pi\rho(x, y, z)$:

$$u(M) = \left(\frac{1}{r} * \rho\right) = \iiint\limits_D \frac{\rho(\xi, \eta, \zeta)\, d\xi\, d\eta\, d\zeta}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}}$$

$$= \int\limits_D \frac{\rho(P)\, d\tau_P}{r_{MP}}$$

All the above properties of the volume potential can be deduced immediately from this expression. For example,

$$u = \nabla^2\left(\frac{1}{r} * \rho\right) = \nabla^2\left(\frac{1}{r}\right) * \rho = -4\pi\,\delta(M) * \rho(M) = -4\pi\,\rho(M)$$

since $\nabla^2(1/r) = -4\pi\,\delta(M)$ and $\delta(M) * \rho(M) = \rho(M)$.

## 7.2 POTENTIAL DUE TO A SIMPLE LAYER

Suppose that charges (masses) are distributed on a surface $S$ with density $\rho(P)$. The potential due to these charges is given by

$$v(M) = \int\limits_S \frac{\rho(P)}{r_{MP}}\, d\sigma_P \tag{17}$$

This integral is called the potential due to a simple layer. We shall suppose henceforth that the function $\rho(P)$ is bounded, $|\rho(P)| \leqslant H$, and the surface $S$ is a Lyapunov surface. The surface $S$ is defined as a Lyapunov surface if it has the following properties.

1. The surface $S$ possesses a tangential plane at each point on it.

2. For each point $P$ on $S$ there exists a neighbourhood $S_P$ which is such that any straight line parallel to the normal at $P$ intersects $S_P$ not more than twice.

3. The angle $\gamma(P, P_1) = (\boldsymbol{n}_P, \boldsymbol{n}_{P_1})$ between the normals $\boldsymbol{n}_P$ and $\boldsymbol{n}_{P_1}$ at the points $P$ and $P_1$ satisfies the following condition:

$$\gamma(P, P_1) < Ar_{PP_1}^{\delta}$$

where $A$ and $\delta$ are constants and $0 < \delta \leqslant 1$.

Let us consider some of the properties of the potential due to a simple layer.

*Property 1* The potential of a simple layer is defined everywhere.

For points $M$ not belonging to the surface $S$ this is self-evident. If $M \in S$ the integral given by (17) is an improper integral in the two-dimensional region $S$. It is known that the improper two-dimensional integral

$$\int \frac{\mathrm{d}\sigma_P}{r_{MP}^{\alpha}}$$

converges absolutely if $\alpha < 2$. In our case, $\alpha = 1$ and, consequently, (17) converges.

*Property 2* The potential due to a simple layer is continuous everywhere.

If $M \notin S$, the integral given by (17) is not an improper integral and its continuity follows directly from the continuity of the integrand, $1/r_{MP}$.

If $M_0 \in S$, it is sufficient to show the uniform convergence of (17) in the neighbourhood of $M_0$. Consider the integral

$$v_1(M) = \int\limits_{S_{M_0}^{\delta}} \frac{\rho(P)\,\mathrm{d}\sigma_P}{r_{MP}}$$

taken over the surface $S_{M_0}^{\delta}$ ($S_{M_0}^{\delta} \subset S$) containing the point $M_0$ and having a diameter less than $\delta$, $\mathrm{d}(S_{M_0}^{\delta}) < \delta$. We shall use a set of coordinates with the origin at $M_0$ and the $z$ axis parallel to the normal to $S$ at this point. Let $M(x, y, z)$ be an arbitrary point separated from $M_0$ by a distance less than $\delta(\overline{MM_0} < \delta)$. Let $\Sigma_{M_0}^{\delta}$ represent the projection of the surface $S_{M_0}^{\delta}$ on to the $(x, y)$ plane and let $Q_{M_1}^{2\delta}$ be the circle on the $(x, y)$ plane, centred on the point $M_1(x, y, 0)$ and of radius $2\delta$. It is evident that $\Sigma_{M_0}^{\delta} \subset Q_{M_1}^{2\delta}$. The projection of

a surface element $d\sigma$ on to the $(x, y)$ plane is $ds = d\sigma \cos\gamma$, where $\gamma$ is the angle between the normal to $S$ and the $z$ axis. It is clear that

$$|v_1(M)| \leqslant H \int_{S_{M_0}^{\delta}} \frac{d\sigma_P}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}}$$

$$\leqslant H \int_{S_{M_0}^{\delta}} \frac{d\sigma_P}{\sqrt{(x-\xi)^2+(y-\eta)^2}}$$

$$= H \int_{\Sigma_{M_0}^{\delta}} \frac{ds}{\cos\gamma\sqrt{(x-\xi)^2+(y-\eta)^2}}$$

From the third property of Lyapunov surfaces, $\delta$ can be taken to be so small that for points $P \in S_{M_0}^{\delta}$ we have $\cos\gamma \geqslant 1/2$. We therefore have

$$|v_1(M)| \leqslant 2H \int_{\Sigma_{M_0}^{\delta}} \frac{ds}{\sqrt{(x-\xi)^2+(y-\eta)^2}}$$

$$\leqslant 2H \int_{Q_{M_1}^{2\delta}} \frac{ds}{\sqrt{(x-\xi)^2+(y-\eta)^2}}$$

If we now introduce a polar set of coordinates with the origin at $M_1$, we can readily evaluate the last integral. It is equal to

$$2H \int_{Q_{M_1}^{2\delta}} \frac{ds}{r} = 2H \int_0^{2\delta}\int_0^{2\pi} dr\, d\varphi = 8\pi H\delta$$

In order that the integral $|v_1(M)|$ be less than the given number $\varepsilon$, it is sufficient to take $\delta < 1/8\pi H$.

*Property 3* The potential due to a simple layer is a harmonic function everywhere except for points on the surface $S$.

This property follows immediately since for points $M \notin S$ the integral (17) is not an improper integral and therefore

$$\nabla^2 v = \int_S \rho(P)\nabla^2\left(\frac{1}{r_{MP}}\right) d\sigma_P \equiv 0$$

*Property 4* If the surface $S$ is bounded, the potential of the simple layer tends to zero as $M$ tends to infinity.

To prove this, let us apply to (17) the mean value theorem:

$$v(M) = \frac{1}{r_{MP^*}} \int_S \rho(P)\, d\sigma_P = \frac{m}{r_{MP^*}} \tag{18}$$

where $P^* \in S$, $m = \int_S \rho\, d\sigma$ is the total charge.

Property 4 follows directly from (18).

*Property 5* The normal derivatives of the potential due to a simple layer have a discontinuity of the first kind at points on the surface $S$, and the magnitude of the discontinuity is $4\pi\rho(M)$.

We shall not reproduce the proof of this property here.

For the two-dimensional case (plane) the potential of a simple layer is of the form

$$v(M) = \int_C \rho(P) \ln\left(\frac{1}{r_{MP}}\right) ds_P$$

Properties 1–3 are valid for this function. As $M$ tends to infinity, $v(M)$ tends to infinity as $\ln r_{MP}$. The discontinuous change in the normal derivatives at points on the curve $C$ is $2\pi\rho(M)$. The proofs of all these properties are similar to those given for the three-dimensional case and will not be reproduced here.

## 7.3 POTENTIAL DUE TO A DOUBLE LAYER

**7.3.1** Consider charges $-e$ and $e$ at points $P_1$ and $P_2$, respectively (Fig. 7.1). The electrostatic potential due to this dipole is given by

$$w(M) = e\left(\frac{1}{r_{MP_2}} - \frac{1}{r_{MP_1}}\right)$$

or

$$w(M) = eh\frac{\partial}{\partial n}\left(\frac{1}{r_{MP}}\right)\bigg|_{P=P^*}$$

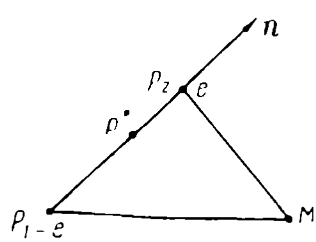



*Fig. 7.1*

where $P^*$ is a point on the segment $P_1P_2$ and the derivative is evaluated in the direction $\boldsymbol{n}$ of the segment drawn from $P_1$ to $P_2$ (axis of the dipole). The quantity $h$ is the distance between $P_1$ and $P_2$. The dipole moment is defined by $eh = \nu$. If we allow the points $P_1$ and $P_2$ to approach each other at constant $\nu$ (by increasing the magnitude of the charges $e$), then in the limit as $h \to 0$ we obtain a point dipole located at $P$. The potential due to this dipole is given by

$$w(M) = \nu\frac{\partial}{\partial n}\left(\frac{1}{r_{MP}}\right)$$

where the derivative is evaluated with respect to the coordinates of $P$ and in the direction of the dipole axis.

Let $S$ be a two-sided surface with continuously varying tangential plane. This means that if at some point $P$ on the surface we choose the positive direction of the normal $\boldsymbol{n}_P$ to the surface and allow $P$ to move along a closed curve (lying on $S$), then if the direction of the normal varies continuously, as we return to the initial point the direction of the normal will become the same as the original direction. One of the directions of the normal to the surface can be taken as positive since the unit vector in this direction will be continuous on the surface. We shall assume that this positive direction has been chosen.

**7.3.2** If on a two-sided surface $S$ there is a distribution of dipoles with dipole moment density $\nu(P)$ such that the axes of the dipoles at each point coincide with the positive direction of the normal, then the potential due to these dipoles is given by

$$w(M) = \int_S \nu(P)\frac{\partial}{\partial n}\left(\frac{1}{r_{MP}}\right) d\sigma_P \tag{19}$$

Let $S$ be a two-sided surface with chosen positive direction of the normal. Let us imagine that we have drawn lines of length $h$

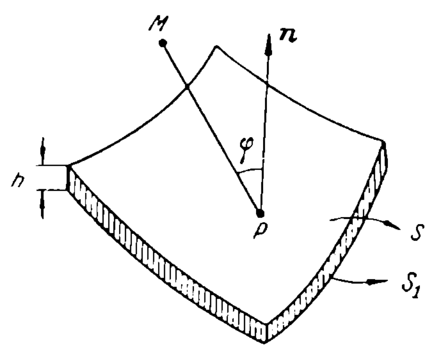



*Fig. 7.2*

along the positive direction of the normal at each point. The geometric locus of the end-points of these lines is a surface $S_1$ which is separated from $S$ by the distance $h$. Suppose that on $S$ there are negative charges distributed with density $\nu(P)/h$ and on $S_1$ there are positive charges with the same density (Fig. 7.2).

We thus have a double layer of charges of opposite signs, which can be regarded as a set of dipoles distributed over $S$ and $S_1$ with the density $\nu(P)/h$. The potential due to the dipole on an element $\mathrm{d}\sigma$ of $S$ and $S_1$ is $\nu(P)(\partial/\partial n)(1/r_{MP})\mathrm{d}\sigma$. The potential due to all the dipoles is therefore given by

$$\int_S \nu(P)\frac{\partial}{\partial n}\left(\frac{1}{r_{MP}}\right)\mathrm{d}\sigma_P$$

If we let $h$ tend to zero, we obtain a double layer on $S$, whose potential is given by (19).

Since

$$\frac{\partial}{\partial n}\left(\frac{1}{r_{MP}}\right) = \frac{\cos\varphi}{r_{MP}^2}$$

where $\varphi$ is the angle between the positive direction of the normal to $S$ at $P$ and the line $PM$, the potential due to a double layer can also be written in the form

$$w(M) = \int_S \nu(P)\frac{\cos\varphi}{r_{MP}^2}\,\mathrm{d}\sigma_P \tag{20}$$

Let $\mathrm{d}\omega_{MP}$ be the solid angle which $\mathrm{d}\sigma_P$ subtends at $M$ so that

$$r_{MP}^2\,\mathrm{d}\omega_{MP} = \cos\varphi\,\mathrm{d}\sigma_P$$

This formula follows directly from the fact that, by definition, $\mathrm{d}\omega_{MP}$ is equal to the area of an element on a unit sphere centred on $M$ which is cut by the cone with apex at $M$ and base $\mathrm{d}\sigma_P$ (Fig. 7.3). $\mathrm{d}\omega_{MP}$ is positive if the angle $\varphi$ is acute, and negative if it is obtuse.

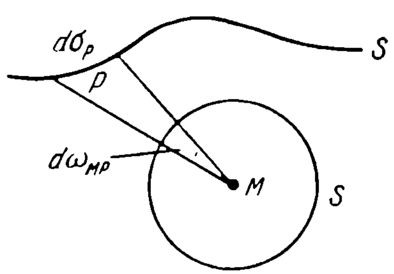



*Fig. 7.3*

The double-layer potential can, therefore, also be written in the form

$$w(M) = \int_S \nu(P)\, d\omega_{MP} \tag{21}$$

**7.3.3** It follows from (21) that the potential due to a double layer is defined also at the points $M$ on the surface $S$. We therefore have the following two properties.

*Property 1* The potential due to a double layer is defined everywhere.

*Property 2* At points $M$ which do not lie on $S$, the potential due to the double layer is a harmonic function.

To show this, we can use Equation (20). If $M \notin S$, the integral given by (20) is not an improper integral and, therefore,

$$\nabla^2 w = \nabla^2 \left( \int_S \nu(P) \frac{\partial}{\partial n}\left(\frac{1}{r_{MP}}\right) d\sigma_P \right) = \int_S \nu(P) \nabla^2 \left\{ \frac{\partial}{\partial n}\left(\frac{1}{r_{MP}}\right) \right\} d\sigma_P$$

$$= \int_S \nu(P) \frac{\partial}{\partial n} \left\{ \nabla^2 \left(\frac{1}{r_{MP}}\right) \right\} d\sigma_P \equiv 0$$

*Property 3* As the point of observation $M$ tends to infinity, the double-layer potential tends to zero. We are assuming that the surface $S$ has a finite area and lies in a finite region.

To prove this, let us use Equation (20). Applying the mean value theorem to this integral, we obtain

$$w(M) = \nu(P) \frac{\cos\varphi}{r_{MP}^2} \bigg|_{P=P^*} \int_S d\sigma_P$$

where $P^* \in S$. The validity of the property follows at once.

We shall assume henceforth that the surface $S$ is closed. The positive direction of the normal will be taken to be the direction of the inward normal to $S$.

Consider the special case of the potential due to a double layer when the dipole moment density is a constant, $\nu_0$. For this potential we have the following results:

$$\tilde{w}(M) = \begin{cases} 4\pi\nu_0, & M \text{ inside } S \\ 2\pi\nu_0, & M \text{ on } S \\ 0, & M \text{ outside } S \end{cases}$$

To prove this, let us use (21). Suppose that $M$ lies inside $S$. Let us suppose to begin with that a ray drawn from $M$ cuts the surface $S$ at one point only. The integral $\int_S d\omega_{MP}$ is then equal to the total solid angle subtended by $S$ at an internal point, i.e. $4\pi$. Consequently, in this case, $\tilde{w}(M) = 4\pi\nu_0$.

If some or all rays drawn from the point $M$ intersect the surface $S$ in a finite number ($\leqslant k$) of points, the solid angles $d\omega_{MP}$ subtended by the surface element $d\sigma_P$ defined by rays drawn from inside $S$ (Fig. 7.4) will be positive, whereas the solid angles $d\omega_{MP}$ subtended

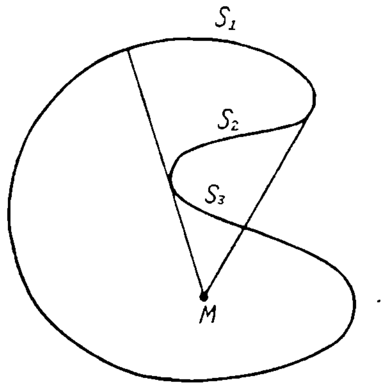



*Fig. 7.4*

by the surface element $d\sigma_P$ defined by rays drawn from outside $S$ will be negative. Hence, the angle $\varphi$ between the inward normal and the direction of ***PM*** will be obtuse and, consequently, cos $\varphi$ will be negative. It is, therefore, evident that

$$\int_{S_3} d\omega_{MP} + \int_{S_2} d\omega_{MP} = 0$$

The algebraic sum of all solid angles $d\omega_{MP}$ will therefore also be equal to $4\pi$.

It follows that, in this case again, $\tilde{w}(M) = 4\pi\nu_0$. If the point $M$ lies outside $S$, the solid angle $d\omega_{MP}$ corresponding to elements $d\sigma_P$ on $S_1$ (Fig. 7.5) will be negative, whereas the solid angles $d\omega_{MP}$ corresponding to elements $d\sigma_P$ on $S_2$ will be positive. Therefore,

$$\int_S d\omega_{MP} = \int_{S_1} d\omega_{MP} + \int_{S_2} d\omega_{MP} = 0$$

It follows that if the point $M$ lies outside $S$, then $\tilde{w}(M) = 0$. Similarly, it can be shown that $\tilde{w}(M) = 2\pi\nu_0$ if $M \in S$.

We can now understand the properties of the potential due to a double layer in the neighbourhood of the point $M$ lying on the surface on which the double layer is situated.

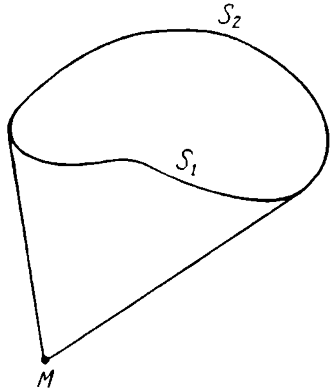



*Fig. 7.5*

*Property 4* If the dipole moment density $\nu(P)$ is continuous on $S$, the potential of the double layer, $w(M)$, will have a discontinuity of the first kind at points on $S$ and the discontinuity is

$$w_{\text{ext}}(M_0) - w_{\text{int}}(M_0) = 4\pi\nu(M_0), \quad M_0 \in S$$

where $w_{\text{ext}}(M_0)$ is the limit of $w(M)$ at $M_0$, when $M$ tends to $M_0$ from outside the surface and $w_{\text{int}}$ $(M_0)$ is the limit of $w(M)$ at $M_0$ when $M$ tends to $M_0$ from inside.

We shall suppose, for simplicity, that each ray drawn from the point $M$ intersects $S$ not more than $k$ times (although this statement is not valid for an arbitrary Lyapunov surface). Let $M_0$ be a fixed point on $S$ and consider the auxiliary function

$$\bar{w}(M) = \int_S \{\nu(P) - \nu(M_0)\} \, d\omega_{MP} = w(M) - \tilde{w}(M) \qquad (22)$$

*Lemma* The function $\bar{w}(M)$ is continuous at $M_0$.

*Proof* Let $S'$ represent the part of $S$ which lies in a $\delta$-neighbourhood $D^{\delta}_{M_0}$ of $M_0$, and let $S''$ represent the remaining part of $S$. We may then write

$$\bar{w}(M) = \bar{w}_1(M) + \bar{w}_2(M)$$

where

$$\bar{w}_1(M) = \int_{S'} \{\nu(P) - \nu(M_0)\} \, d\omega_{MP}$$

$$\bar{w}_2(M) = \int_{S''} \{\nu(P) - \nu(M_0)\} \, d\omega_{MP}$$

The function $\bar{w}_2(M)$ is continuous at $M_0$. Therefore, for arbitrary $\varepsilon > 0$ the quantity $|w_2(M) - w_2(M_0)|$ will be less than $\varepsilon/3$ if $\overline{MM_0}$ is sufficiently small. Next,

$$|\bar{w}_1(M)| = \left|\int_{S'} \{\nu(P) - \nu(M_0)\}\, d\omega_{MP}\right| \leqslant \int_{S'} |\nu(P) - \nu(M_0)|\, |d\omega_{MP}|$$

Since $\nu(P)$ is continuous at $M_0$, the quantity $|\nu(P) - \nu(M_0)|$ is less than $\varepsilon/12\, k\pi$ if $\delta$ (i.e. the radius of the neighbourhood $D^{\delta}_{M_0}$ of $M_0$) is sufficiently small.
Next,

$$\int_{S'} |d\omega_{MP}| < 4\pi k$$

and, consequently,

$$|\bar{w}_1(M)| \leqslant \int_{S'} |\nu(P) - \nu(M_0)|\, |d\omega_{MP}| < \frac{\varepsilon}{3}$$

and

$$|\bar{w}_1(M_0)| < \frac{\varepsilon}{3}$$

Therefore,

$$|\bar{w}(M) - \bar{w}(M_0)| \leqslant |\bar{w}_1(M)| + |\bar{w}_1(M_0)| + |\bar{w}_2(M) - \bar{w}_2(M_0)| < \varepsilon$$

if the point $M$ is sufficiently close to $M_0$. This proves the lemma.

*Proof of Property 4* Let us pass to the limit in Equation (22) by allowing $M$ to approach $M_0$ from outside and inside $S$. This yields

$$\begin{aligned} \bar{w}_{\text{ext}}(M_0) &= w_{\text{ext}}(M_0) - 4\pi\nu(M_0) = \bar{w}(M_0) \\ &= w(M_0) - 2\pi\nu(M_0) = \bar{w}_{\text{int}}(M_0) = w_{\text{int}}(M_0) - 0 \end{aligned}$$

where $\bar{w}(M_0)$ and $w(M_0)$ are the values of $\bar{w}(M)$ and $w(M)$ at $M_0$ on $S$. From these equations we have

$$w_{\text{ext}}(M_0) = w(M_0) + 2\pi\nu(M_0) \tag{23}$$

$$w_{\text{int}}(M_0) = w(M_0) - 2\pi\nu(M_0) \tag{24}$$

$$w_{\text{ext}}(M_0) - w_{\text{int}}(M_0) = 4\pi\nu(M_0) \tag{25}$$

For the two-dimensional case it can be shown, in a similar way, that at points $M_0$ on the carrier curve $C$ we have

$$w_{ext}(M_0) = w(M_0)+\pi\nu(M_0) \tag{26}$$

$$w_{int}(M_0) = w(M_0)-\pi\nu(M_0) \tag{27}$$

$$w_{ext}(M_0)-w_{int}(M_0) = 2\pi\nu(M_0) \tag{28}$$

## 7.4 APPLICATION OF POTENTIALS TO THE SOLUTION OF BOUNDARY-VALUE PROBLEMS

The above properties of potentials can be used for the solution of boundary-value problems. We shall illustrate this by considering the first internal boundary-value problem

$$\nabla^2 u = f(M) \text{ at } D \tag{29}$$

$$u|_S = \varphi(M) \tag{30}$$

where $u(M)$ is continuous in $\overline{D} = D+S$.

A special solution of (29) is the volume potential (in view of Property 4 in Section 7.1)

$$u_1(M) = -\frac{1}{4\pi}\int\limits_D \frac{f(P)}{r_{MP}}\,d\tau_P$$

It is therefore natural to seek the solution of (29)–(30) in the form of the sum

$$u(M) = u_1(M)+u_2(M)$$

where the boundary-value problem for the function $u_2(M)$ will be formulated as follows:

$$\nabla^2 u_2 = 0 \tag{31}$$

$$u_2|_S = \varphi(M)-u_1(M)|_S = F(M) \tag{32}$$

We shall seek the solution of this problem in the form of a double-layer potential

$$w(M) = \int\limits_S \nu(P)\frac{\cos\varphi}{r_{MP}^2}\,d\sigma_P$$

with suitably chosen $\nu(P)$. For any $\nu(P)$ this potential is harmonic in $D$ (in view of Property 2, Section 7.3). To satisfy the boundary condition (32), it is necessary that the relationship $w_{ext}(M) = F(M)$

should be satisfied at points $M \in S$. Using Equation (23) of Section 7.3, this condition can be written in the form

$$w(M)+2\pi\nu(M) = F(M)$$

or

$$\int_S \nu(P)\frac{\partial}{\partial n}\left(\frac{1}{r_{MP}}\right) d\sigma_P + 2\pi\nu(M) = F(M) \tag{33}$$

The solution of the boundary-value problem (31)–(32) will be the double-layer potential with dipole moment density $\nu(P)$ satisfying the condition given by (33).

It follows that the boundary-value problem under consideration reduces to the solution of the integral equation (33) for $\nu(P)$.

*Example* Let us solve the first boundary-value problem for a circular region of radius $R$ bounded by the circle $C$:

$$\nabla^2 u = 0, \qquad u|_C = F(s)$$

where $s$ is the length of arc on the circle.

At points $M$ on the circle (Fig. 21)

$$\frac{\cos\varphi}{r_{MP}} = \frac{1}{2R} \tag{34}$$

We shall seek the solution in the form of the double-layer potential

$$u(s) = \int_C \nu(\xi)\frac{\partial}{\partial n}\left(\frac{1}{r}\right) d\xi = \int_C \nu(\xi)\frac{\cos\varphi}{r}\, d\xi$$

Using (34), the boundary condition leads to an integral equation for $\nu(s)$:

$$\int_C \frac{1}{2R}\nu(\xi)\, d\xi + \pi\nu(s) = F(s) \tag{35}$$

We shall seek the solution of this equation in the form

$$\nu(s) = \frac{1}{\pi}F(s) + A$$

where $A$ is an unknown constant. Substituting this function into (35) we obtain

$$\int_C \frac{1}{2R}\left[\frac{1}{\pi}F(\xi)+A\right] d\xi + F(s) + \pi A = F(s)$$

and hence

$$\frac{1}{2\pi R}\int\limits_C F(\xi)\,d\xi+2\pi A=0 \quad \text{and} \quad A=-\frac{1}{4\pi^2 R}\int\limits_C F(\xi)\,d\xi$$

It follows that

$$\nu(s)=\frac{1}{\pi}F(s)-\frac{1}{4\pi^2 R}\int\limits_C F(\xi)\,d\xi$$

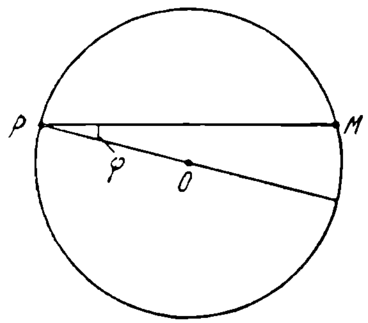



Fig. 7.6

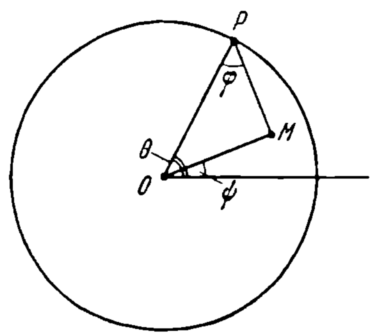



Fig. 7.7

Consequently, the solution of the boundary-value problem is of the form

$$\begin{aligned} u(M) &= \int\limits_C \nu(\xi)\frac{\cos\varphi}{r}\,d\xi \\ &= \frac{1}{\pi}\int\limits_C \frac{F(\xi)\cos\varphi}{r}\,d\xi-\frac{1}{4\pi^2 R}\int\limits_C F(\xi)\,d\xi\int\limits_C \frac{\cos\varphi}{r}\,d\xi \\ &= \frac{1}{\pi}\int\limits_C \frac{\cos\varphi}{r}F(\xi)\,d\xi-\frac{1}{2\pi R}\int\limits_C F(\xi)\,d\xi \\ &= \frac{1}{\pi}\int\limits_C \left(\frac{\cos\varphi}{r}-\frac{1}{2R}\right)F(\xi)\,d\xi \end{aligned}$$

or

$$u=\frac{1}{\pi}\int\limits_C \frac{2rR\cos\varphi-r^2}{2Rr^2}F(\xi)\,d\xi$$

From the triangle $OPM$ (Fig. 7.7) we find ($OM = \rho$):

$$\frac{2Rr\cos\varphi - r^2}{2Rr^2} = \frac{R^2-\rho^2}{2R[R^2+\rho^2-2R\rho\cos(\theta-\psi)]}$$

and therefore

$$u(M) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-\rho^2)F(R\,\theta)\,\mathrm{d}\theta}{R^2+\rho^2-2R\rho\cos(\theta-\psi)}$$

We have thus obtained Poisson's integral.

The solution of the second boundary-value problem for the Laplace equation can be sought in the form of a potential due to a simple layer with unknown density $\rho(P)$.

## PROBLEMS

1. Find the volume potential due to the mass distribution of density $\rho(r)$ in the spherical layer $R_1 \leqslant r \leqslant R_2$. Consider also the case $\rho(r) = \rho_0 = \text{const}$.

2. Find the potential due to a spherical simple layer of constant surface charge density $\rho_0$.

3. Find the electrostatic field due to charges distributed uniformly inside a sphere placed above a perfectly conducting plane $z = 0$.

4. Find the potential due to a circle of constant line density of charge.

5. Find the potential due to a simple layer in the form of a segment of constant charge density.

6. Find the potential due to a double layer in the form of a segment of constant dipole-moment density.

7. Use the double-layer potential to solve the first boundary-value problem for the Laplace equation (a) outside the circle and (b) on a half-plane.

# 8

# Integral Equations

We have seen (Section 7.4) that the first boundary-value problem for elliptic equations can be reduced to a linear integral equation. In this chapter we shall discuss some of the basic properties of linear integral equations of the second kind. For the sake of simplicity, we shall consider only the one-dimensional case (except for Section 8.1). All the results will also be valid for the multi-dimensional case.

## 8.1 CLASSIFICATION OF LINEAR INTEGRAL EQUATIONS

Equations of the form

$$\varphi(M)-\lambda \int_D K(M, P)\varphi(P)\,\mathrm{d}\tau_P = f(M)$$

where $\varphi(P)$ is the required function, $f(M)$ and $K(M, P)$ are unknown functions, $D$ is a fixed domain and $\lambda$ is a numerical parameter, are called Fredholm's integral equations of the second kind. When $f(M) \equiv 0$, the equation is called homogeneous; in all other cases it is referred to as inhomogeneous.

Equations of the form

$$\varphi(M)-\lambda \int_{D(M)} K(M, P)\varphi(P)\,\mathrm{d}\tau_P = f(M)$$

where $D(M)$ is a variable domain which depends on the point $M$,

are called Volterra integral equations of the second kind. For example, in the one-dimensional case

$$\varphi(x)-\lambda\int_a^x K(x, s)\varphi(s)\,\mathrm{d}s = f(x)$$

When $f(M) \equiv 0$, the equation is called homogeneous; otherwise it is referred to as inhomogeneous. Equations of the form

$$\int_D K(M, P)\varphi(P)\,\mathrm{d}\tau_P = f(M)$$

where $D$ is a fixed domain, are called Fredholm's integral equations of the first kind.

Equations of the form

$$\int_{D(M)} K(M, P)\varphi(P)\,\mathrm{d}\tau_P = f(M)$$

are called Volterra integral equations of the first kind. The function $K(M, P)$ is called the kernel of the integral equation.

*Remark* Volterra equations are special cases of Fredholm's equations. Thus, if in the one-dimensional case we let

$$K_1(x, s) = \begin{cases} 0, & x < s < b \\ K(x, s), & a < s \leqslant x \end{cases}$$

the Volterra equation .

$$\varphi(x)-\lambda\int_a^x K(x, s)\varphi(s)\,\mathrm{d}s = f(x)$$

can be rewritten as the Fredholm equation with a kernel $K_1(x, s)$:

$$\varphi(x)-\lambda\int_a^b K_1(x, s)\varphi(s)\,\mathrm{d}s = f(x)$$

The kernels $K_1(x, s)$ are called Volterra kernels.

## 8.2 PROBLEMS LEADING TO INTEGRAL EQUATIONS

**8.2.1** Cauchy's problem for a linear ordinary differential equation of order $n$ can be reduced to a Volterra integral equation of the second kind. To be specific, consider the second-order equation

$$y''+a(x)y'+b(x)y = f(x) \tag{1}$$

$$y(0) = y_0, \qquad y'(0) = y_0' \tag{2}$$

Let

$$y''(x) = \varphi(x) \tag{3}$$

We then have

$$y'(x) = y'_0 + \int_0^x \varphi(s)\, ds \tag{4}$$

$$y(x) = y_0 + \int_0^x y'(\xi)\, d\xi \tag{5}$$

Substituting (4) into (5) we obtain

$$y(x) = y_0 + xy'_0 + \int_0^x \int_0^\xi \varphi(s)\, ds\, d\xi$$

If we rearrange the order of integration in this expression we obtain

$$y(x) = y_0 + xy'_0 + \int_0^x (x-s)\varphi(s)\, ds \tag{6}$$

We have thus expressed the function $y(x)$ and its derivatives $y'(x)$ and $y''(x)$ in terms of the function $\varphi(x)$ defined by (3), (4) and (6). Substituting these values into (1) and taking the functions $a(x)$ and $b(x)$ under the integral sign, we obtain the following Volterra integral equation of the second kind:

$$\varphi(x) + \int_0^x \{a(x)+(x-s)b(x)\}\varphi(s)\, ds = f_1(x) \tag{7}$$

with the kernel $K(x, s) = a(x)+(x-s)b(x)$, where

$$f_1(x) = f(x) - y'_0 a(x) - y_0 b(x) - xy'_0 b(x)$$

The procedure for reducing the Cauchy problem to an integral equation for the equations of the $n$-th order is quite similar.

**8.2.2** We shall now show that the Sturm–Liouville problem on a finite segment can be reduced to a Fredholm integral equation of the second kind. To do this, we shall introduce Green's function for the boundary-value problem and investigate its simplest properties.

*Definition* Green's function $G(x, s)$ for the boundary-value problem

$$L[y] \equiv \frac{d}{dx}[k(x)y'] - q(x)y = -f(x)$$

$$\alpha_1 y'(0) - \beta_1 y(0) = 0, \qquad \alpha_2 y'(l) + \beta_2 y(l) = 0$$

is the solution of the boundary-value problem

$$L[y] = -\delta(x-s)$$
$$\alpha_1 y'(0)-\beta_1 y(0) = 0, \qquad \alpha_2 y'(l)+\beta_2 y(l) = 0$$

which is continuous in the segment $[0, l]$. We shall now summarise some of the properties of Green's function $G(x, s)$.

1. Green's function is symmetric, i.e.

$$G(x, s) = G(s, x)$$

*Proof* Let us apply Green's formula to the one-dimensional case (Section 6.1) to the functions $v = G_1 = G(x, s_1)$ and $u = G_2 = G(x, s_2)$. We have

$$\int_0^l \{G(x, s_2)\delta(x-s_1)-G(x, s_1)\delta(x-s_2)\}\,dx = k(x)\left(G_1\frac{\partial G_2}{\partial x}-G_2\frac{\partial G_1}{\partial x}\right)\Bigg|_0^l \tag{8}$$

Using the properties of the $\delta$ function, the integral on the left-hand side of Equation (8) is equal to $G(s_1, s_2)-G(s_2, s_1)$, whereas the right-hand side is zero. For the first and second boundary-value problems this follows directly from the fact that the functions $G_1$ and $G_2$, or the derivatives $\partial G_1/\partial x$ and $\partial G_2/\partial x$, vanish at the ends of the segment ($x = 0$ and $x = l$). For the third boundary-value problem we can express the derivatives $\partial G_1/\partial x$ and $\partial G_2/\partial x$ at the end of the segment in terms of $G_1$ and $G_2$:

$$\frac{\partial G_1}{\partial x}\bigg|_{x=0} = \frac{\beta_1}{\alpha_1}G(0, s_1), \qquad \frac{\partial G_2}{\partial x}\bigg|_{x=0} = \frac{\beta_1}{\alpha_1}G(0, s_2)$$
$$\frac{\partial G_1}{\partial x}\bigg|_{x=l} = \frac{-\beta_2}{\alpha_2}G(l, s_1), \qquad \frac{\partial G_2}{\partial x}\bigg|_{x=l} = \frac{-\beta_2}{\alpha_2}G(l, s_2)$$

and substitute these values into the right-hand side of (8). We obtain

$$-\frac{\beta_2}{\alpha_2}G(l, s_1)G(l, s_2)+\frac{\beta_2}{\alpha_2}G(l, s_2)G(l, s_1)$$
$$+\frac{\beta_1}{\alpha_1}G(0, s_1)G(0, s_2)-\frac{\beta_1}{\alpha_1}G(0, s_2)G(0, s_1) = 0$$

It follows that, in fact,

$$G(s_2, s_1) = G(s_1, s_2)$$

2. The partial derivative of Green's function $G_x(x, s)$ has a discontinuity of the first kind at $x = s$ and the magnitude of this discontinuity is

$$G_x(s+0, s) - G_x(s-0, s) = -\frac{1}{k(s)} \tag{9}$$

To prove this, let us integrate the identity

$$L[G] \equiv -\delta(x-s)$$

with respect to $x$ between $s-\varepsilon$ and $s+\varepsilon$, where $\varepsilon > 0$. We obtain

$$\int_{s-\varepsilon}^{s+\varepsilon} L[G]\,\mathrm{d}x = \int_{s-\varepsilon}^{s+\varepsilon} \left\{\frac{\mathrm{d}}{\mathrm{d}x}[k(x)G_x(x, s)] - q(x)G(x, s)\right\}\mathrm{d}x = -1$$

or

$$k(x)G_x(x, s)\Big|_{s-\varepsilon}^{s+\varepsilon} - \int_{s-\varepsilon}^{s+\varepsilon} q(x)G(x, s)\,\mathrm{d}x = -1$$

On passing to the limit in this equation as $\varepsilon \to 0$, we obtain

$$G_x(s+0, s) - G_x(s-0, s) = -\frac{1}{k(s)}$$

since $\lim\limits_{\varepsilon\to 0} \int_{s-\varepsilon}^{s+\varepsilon} q(x)G(x, s)\,\mathrm{d}x = 0$.

**8.2.3** *Theorem* The Green's function is unique.

To begin with, let us establish the following two lemmas.

*Lemma 1* There exists a solution $y_1(x)$ of the equation $L[y] = 0$, which satisfies the boundary condition $\alpha_1 y'(0) - \beta_1 y(0) = 0$.

*Proof* It is known that Cauchy's problem for the equation $L[y] = 0$ with any initial values $y(0) = y_0$, $y'(0) = y'_0$ has a solution. It follows, in particular, that there exists a solution with initial values $y(0)$ and $y'(0)$ which are related by $\alpha_1 y'(0) - \beta_1 y(0) = 0$. This proves the lemma.

*Lemma 2* Any two solutions $y_1(x)$ and $\bar{y}_1(x)$ of the equation $L[y] = 0$ which satisfy the same boundary condition differ from each other only by a constant factor, i.e. $\bar{y}_1(x) = C_1 y_1(x)$.

*Proof* The functions $y_1(x)$ and $\bar{y}_1(x)$ are solutions of the second-order linear equation $L[y] = 0$ and satisfy the conditions

$$\begin{aligned} \alpha_1 y_1'(0) - \beta_1 y_1(0) &= 0 \\ \alpha_1 \bar{y}_1'(0) - \beta_1 \bar{y}_1(0) &= 0 \end{aligned} \tag{10}$$

These relationships can be regarded as a system of equations for $\alpha_1$ and $\beta_1$. Since at least one of the numbers $\alpha_1$ and $\beta_1$ is not zero, the determinant of the system (10) is zero:

$$w(0) = \begin{vmatrix} y_1'(0) & y_1(0) \\ \bar{y}_1'(0) & \bar{y}_1(0) \end{vmatrix} = 0$$

This determinant is the value of the Wronskian at $x = 0$ for the solutions $y_1(x)$ and $y_1(\bar{x})$. It is known that the Wronskian made up of the solutions of a given linear homogeneous equation either is identically zero, or is everywhere non-zero. Since in our case $w(0) = 0$, the Wronskian for $y_1(x)$ and $\bar{y}_1(x)$ is identically zero. Hence, it follows that $\bar{y}_1(x) = Cy_1(x)$.

We shall now prove the above theorem. We shall suppose that $\lambda = 0$ is not an eigenvalue of the boundary-value problem

$$\begin{gathered} L[y] + \lambda y = 0 \\ \alpha_1 y'(0) - \beta_1 y(0) = 0, \quad \alpha_2 y'(l) + \beta_2 y(l) = 0 \end{gathered} \tag{10a}$$

Let $y_1(x)$ be the solution of the equation $L[y] = 0$ which satisfies the boundary condition $\alpha_1 y'(0) - \beta_1 y(0) = 0$. This solution exists in view of Lemma 1. Any other solution satisfying the same boundary condition will be of the form $C_1 y_1(x)$, in accordance with Lemma 2. All this shows that there exists the solution $y_2(x)$ of the equation $L[y] = 0$ which satisfies the boundary condition $\alpha_2 y'(l) + \beta_2 y(l) = 0$. Any solution of $L[y] = 0$ which satisfies the same boundary condition must be of the form $C_2 y_2(x)$.

The functions $y_1(x)$ and $y_2(x)$ are linearly independent. If this were not so, we would have $y_2(x) \equiv Cy_1(x)$. However, the function $y_2(x)$ would then be a solution of $L[y] = 0$, satisfying both boundary conditions. Consequently, $\lambda = 0$ would be an eigenvalue of the boundary-value problem (10a), which contradicts the original assumption.

By choosing $C_1$ and $C_2$ in a suitable way, we can construct the Green's function from the functions $C_1 y_1(x)$ and $C_2 y_2(x)$. Let

$$G(x, s) = \begin{cases} C_1 y_1(x), & x \leqslant s \\ C_2 y_2(x), & x \geqslant s \end{cases} \tag{11}$$

Since Green's function is continuous at $x = s$, we have

$$C_1 y_1(s) = C_2 y_2(s)$$

Hence,

$$\frac{C_1}{y_2(s)} = \frac{C_2}{y_1(s)} = C$$

Consequently, $C_1 = Cy_2(s)$, $C_2 = Cy_1(s)$. The coefficient $C$ can be determined from (9) which must be satisfied by Green's function:

$$C[y_1(s)y_2'(s) - y_2(s)y_1'(s)] = -\frac{1}{k(s)} \tag{12}$$

The expression in the square brackets is the Wronskian of the solutions $y_1(x)$ and $y_2(x)$ and is equal to $D/k(s)$ ($D =$ const). Since the functions $y_1(x)$ and $y_2(x)$ are defined to within constant factors, they can be chosen so that the Wronskian of the solutions $y_1(s)$ and $y_2(s)$ is equal to $-1/k(s)$, i.e. we can suppose that $D = -1$. The relationship given by (12) then assumes the form

$$-\frac{C}{k(s)} = -\frac{1}{k(s)}$$

and hence $C = 1$. Green's function is therefore of the form

$$G(x, s) = \begin{cases} y_2(s)y_1(x), & x \leqslant s \\ y_1(s)y_2(x), & x \geqslant s \end{cases} \tag{13}$$

From (9) and (13) it follows directly that

$$G_x(x, x-0) - G_x(x, x+0) = -\frac{1}{k(x)} \tag{14}$$

**8.2.4** We shall now prove two theorems due to Hilbert.

*Hilbert's first theorem* For any integrable function $f(x)$, the solution $y(x)$ of the boundary-value problem

$$L[y] = -f(x) \tag{15}$$

$$\alpha_1 y'(0) - \beta_1 y(0) = 0, \qquad \alpha_2 y'(l) + \beta_2 y(l) = 0 \tag{16}$$

can be represented by the formula

$$y(x) = \int_0^l G(x, \xi) f(\xi)\, d\xi \tag{17}$$

*Proof* Let us apply Green's formula for the one-dimensional case (Section 6.1) to the functions $u = y(x)$ and $v = G(x, s)$. We obtain

$$\int_0^l \{G(x, s)L[y] - y(x)L[G]\}\, dx = k(x)[G(x, s)y'(x) - y(x)G_x(x, s)]_0^l$$

or

$$-\int_0^l G(x, s)f(x)\, dx + \int_0^l y(x)\delta(x-s)\, dx = k(x)[G(x, s)y'(x) - y(x)G_x(x, s)]_0^l$$

From the boundary conditions (16) for $y(x)$ and $G(x, s)$ it follows that the left-hand side of this equation is zero. Consequently,

$$\int_0^l G(x, s)\, f(x)\, dx = y(s)$$

Changing the symbol used for the integration variable and using the symmetry of Green's function, we obtain the formula stated in the theorem.

*Hilbert's second theorem* For any integrable function $f(x)$, the function

$$y(x) = \int_0^l G(x, \xi)f(\xi)\, d\xi \tag{18}$$

is a solution of the boundary-value problem (15)–(16).

*Proof* It is clear that the function $y(x)$ is continuous in the range $[0, l]$ and

$$y'(x) = \int_0^l G_x(x, \xi)f(\xi)\, d\xi \tag{19}$$

It follows that

$$\alpha_1 y'(0) - \beta_1 y(0) = \int_0^l \{\alpha_1 G_x(0, \xi) - \beta_1 G(0, \xi)\} f(\xi)\, d\xi = 0$$

since, by definition of Green's function, the integrand is identically zero. Similarly,

$$\alpha_2 y'(l)+\beta_2 y(l) = 0 \tag{20}$$

It follows that the function $y(x)$ satisfies the boundary conditions (16).

Let us evaluate $L[y]$. We have

$$L[y] \equiv \int_0^l L[G]f(\xi)\,\mathrm{d}\xi \equiv -\int_0^l \delta(x-\xi)f(\xi)\,\mathrm{d}\xi \equiv -f(x)$$

It follows that $y(x)$ satisfies (15) and this proves the theorem.

**8.2.5** Let us apply Hilbert's theorem to the Sturm–Liouville problem

$$L[y]+\lambda\rho(x)y = 0 \tag{21}$$

$$\alpha_1 y'(0)-\beta_1 y(0) = 0, \qquad \alpha_2 y'(l)+\beta_2 y(l) = 0 \tag{22}$$

From Hilbert's first theorem the solution of this problem is given by

$$y(x) = \lambda\int_0^l G(x, s)\rho(s)y(s)\,\mathrm{d}s \tag{23}$$

i.e. the required solution is a Fredholm integral equation. Conversely, by Hilbert's second theorem, (23) is a solution of the Sturm–Liouville problem (21)–(22). It follows that the Sturm–Liouville problem is equivalent to the integral equation (23).

We have considered the case when $\lambda = 0$ was not an eigenvalue of the boundary-value problem (10a). When $\lambda = 0$ is an eigenvalue of (10a), Green's function $F(x, s)$ for the boundary-value problem

$$L[y] = -f(x)$$

$$\alpha_1 y'(0)-\beta_1 y(0) = 0, \qquad \alpha_2 y'(l)+\beta_2 y(l) = 0$$

will be defined as the solution of the boundary-value problem

$$L[y] = -\delta(x-s)+\Phi_0(x)\Phi_0(s)$$

$$\alpha_1 y'(0)-\beta_1 y(0) = 0, \qquad \alpha_2 y'(l)+\beta_2 y(l) = 0$$

which is continuous in $[0, l]$ and is orthogonal to the eigenfunction $\Phi_0(x)$ corresponding to the eigenvalue $\lambda = 0$, i.e. such that

$$\int_0^l \Phi_0(x) G(x, s)\, dx = 0$$

The corresponding properties of Green's function, in this case, can be established in a similar way.

## 8.3 INTEGRAL EQUATIONS WITH DEGENERATE KERNELS

A kernel $K(x, s)$ is called degenerate if it is of the form

$$K(x, s) = \sum_{i=1}^{n} a_i(x) b_i(s) \tag{24}$$

where $a_i(x)$ are linearly independent functions.

The solution of Fredholm's equation of the second kind with a degenerate kernel can be reduced to the solution of a system of linear algebraic equations.

In point of fact, substituting (24) into

$$\varphi(x) = \lambda \int_a^b K(x, s)\varphi(s)\, ds + f(x) \tag{25}$$

we obtain

$$\varphi(x) = \lambda \sum_{i=1}^{n} C_i a_i(x) + f(x) \tag{26}$$

where $C_i = \int_a^b b_i(s)\varphi(s)\, ds$ are unknown numbers.

It follows that the solution of (25) with a degenerate kernel must be sought in the form given by (26). Substituting this function into (25) and comparing the coefficients of the same functions $a_i(x)$, we obtain the following system of linear algebraic equations

$$C_i = \lambda \sum_{j=1}^{n} C_j \alpha_{ij} + \beta_i \qquad (i = 1, 2, \ldots, n) \tag{27}$$

where

$$\alpha_{ij} = \int_a^b a_i(s) b_j(s)\, ds, \qquad \beta_i = \int_a^b f(s) b_i(s)\, ds$$

By solving this system we can find $C_i$ and, consequently, the solution of (25).

## 8.4 EXISTENCE OF SOLUTIONS

**8.4.1** If the kernel is degenerate, then the existence of the solution of the Fredholm integral equation reduces to the question of the existence of the solution of the corresponding system of algebraic equations (27). We shall prove the existence of the solution of (25) (for sufficiently small values of $|\lambda|$) by the method of successive approximations in a more general case.

We shall assume, for simplicity, that (1) the kernel $K(x, s)$ is continuous for $a \leqslant x, s \leqslant b$, in which case it is bounded by a certain constant $A$, $|K| \leqslant A$, and (2) the function $f(x)$ is continuous in the range $[a, b]$ so that it is bounded in this range by a certain constant $B$, $|f| \leqslant B$. Consider the sequence of functions

$$\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x), \ldots$$

such that

$$\varphi_1(x) = f(x) + \lambda \int_a^b K(x, s) \varphi_0(s)\, ds \tag{28}$$

where $\varphi_0(s)$ is an arbitrary fixed continuous function

$$\varphi_2(x) = f(x) + \lambda \int_a^b K(x, s) \varphi_1(s)\, ds \tag{29}$$

. . . . . . . . . . . . . . . . .

$$\varphi_n(x) = f(x) + \lambda \int_a^b K(x, s) \varphi_{n-1}(s)\, ds \tag{30}$$

. . . . . . . . . . . . . . . . . .

*Theorem* The sequence of functions given by (28)–(30) converges uniformly in the range $[a, b]$ to the function $\bar{\varphi}(x)$ which is a solution of (25).

*Proof* Substituting $\varphi_1(x)$ into the formula for $\varphi_2(x)$ we obtain

$$\varphi_2(x) = f(x) + \lambda \int_a^b K(x, s) f(s)\, ds + \lambda^2 \int_a^b K(x, s) \int_a^b K(s, t) \varphi_0(t)\, dt\, ds$$

If we change the order of integration in the last integral, we obtain

$$\varphi_2(x) = f(x) + \lambda \int_a^b K_1(x, s) f(s)\, ds + \lambda^2 \int_a^b K_2(x, t) \varphi_0(t)\, dt$$

where

$$K_1(x, s) = K(x, s), \qquad K_2(x, t) = \int_a^b K_1(x, s) K_1(s, t)\, ds$$

Similarly, we find that

$$\varphi_n(x) = f(x) + \lambda \int_a^b K_1(x, s) f(s)\, ds + \lambda^2 \int_a^b K_2(x, s) f(s)\, ds + \dots$$

$$+ \lambda^{n-1} \int_a^b K_{n-1}(x, s) f(s) ds + \lambda^n \int_a^b K_n(x, t) \varphi_0(t)\, dt$$

where $K_n(x, t) = \int_b^a K_1(x, s) K_{n-1}(s, t)\, ds$. The limit of the function $\varphi_n(x)$, if it exists, is equal to the sum of the series

$$\bar{\varphi}(x) = f(x) + \lambda \int_a^b K_1(x, s) f(s)\, ds + \dots + \lambda^n \int_a^b K_n(x, s) f(s)\, ds + \dots \tag{31}$$

We shall now prove the uniform convergence of this series. Consider the integral

$$\int_a^b K_n(x, s) f(s)\, ds$$

It is evident that

$$|K_2(x, s)| \leqslant \int_a^b |K_1(x, s) K_1(s, t)|\, ds \leqslant A^2(b-a)$$

$$|K_3(x, s)| \leqslant \int_a^b |K_1(x, s) K_2(s, t)|\, ds \leqslant A^3(b-a)^2$$

. . . . . . . . . . . . . . . . . . . . . .

$$|K_n(x, s)| \leqslant \int_a^b |K_1(x, s) K_{n-1}(s, t)|\, ds \leqslant A^n(b-a)^{n-1}$$

. . . . . . . . . . . . . . . . . . . . . . .

and, therefore,

$$\left|\int_a^b K_n(x, s)f(s)\,\mathrm{d}s\right| \leqslant A^n(b-a)^{n-1}\int_a^b |f(s)|\,\mathrm{d}s \leqslant A^n B(b-a)^n$$

Consequently, the numerical series

$$\sum_{n=1}^{\infty} BA^n|\lambda|^n(b-a)^n \tag{32}$$

is the majorant series for (31). If $|\lambda| < 1/A(b-a)$, then the series (32) will converge. Consequently, for such $\lambda$ the series (31), and together with it the sequence of functions $\varphi_n(x)$, will converge uniformly to the function $\bar{\varphi}(x)$. This function is a solution of (25). In the limit, as $n \to \infty$ in (30), we obtain

$$\bar{\varphi}(x) \equiv \lambda\int_a^b K(x, s)\bar{\varphi}(s)\,\mathrm{d}s+f(x)$$

The transition to the limit under the integral sign is legitimate since the sequence converges uniformly. We note that the limit $\lim\limits_{n\to\infty} \varphi_n(x) = \bar{\varphi}(x)$ is independent of the choice of $\varphi_0(s)$ (zero-order approximation). The uniqueness of the solution of (25) readily follows from this. In point of fact, if there exists a further solution $\psi(x)$ of (25), then, assuming $\varphi_0(x) = \psi(x)$ in the procedure for constructing the functions (28)–(30), we obtain

$$\varphi_1(x) = \psi(x), \qquad \varphi_2(x) = \psi(x), \quad \ldots, \quad \varphi_n(x) = \psi(x), \ \ldots$$

This sequence has the function $\bar{\varphi}(x)$ as the limit. However, it is also evident that

$$\lim_{n\to\infty} \varphi_n(x) = \psi(x)$$

and, therefore,

$$\bar{\varphi}(x) = \psi(x)$$

**8.4.2** Since the series given by (31) converges for $|\lambda| < 1/A(b-a)$, it follows that for these values of $\lambda$, the series $\sum\limits_{n=1}^{\infty} A^n|\lambda|^{n-1}(b-a)^{n-1}$ will also converge. However, this series is the majorant series for

$\sum_{n=1}^{\infty} \lambda^{n-1}K_n(x, s)$ so that $\sum_{n=1}^{\infty} \lambda^{n-1}K_n(x, s)$ converges uniformly. The series given by (31) can therefore be written in the form

$$\bar{\varphi}(x) = f(x)+\lambda \int_a^b \left\{ \sum_{n=1}^{\infty} \lambda^{n-1} K_n(x, s) \right\} f(s) \, ds$$

or

$$\bar{\varphi}(x) = f(x)+\lambda \int_a^b R(x, s, \lambda) f(s) \, ds \tag{33}$$

where the function $R(x, s, \lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} K_n(x, s)$ is called the resolvent of (25).

It follows that if we know the resolvent of (25), we can use (33) to obtain its solution. We have defined the resolvent only for small values of $|\lambda|$. However, it can also be defined for any finite domain in the complex plane of the variable $\lambda$ by analytical continuation (except, perhaps, for a finite number of singular points in this domain). If this has been done, then (33) yields the solution of (25) for any $\lambda$ except for these singular points. We shall not pause to consider this in any detail.

**8.4.3** If we apply the above procedure to the Volterra equation

$$\varphi(x) = \lambda \int_a^x K(x, s)\varphi(s) \, ds + f(x) \qquad (a \leqslant x \leqslant b) \tag{34}$$

we obtain the following sequence of functions

$$\varphi_1(x) = f(x)+\lambda \int_a^x K(x, s)\varphi_0(s) \, ds$$

$$\varphi_2(x) = f(x)+\lambda \int_a^x K(x, s)\varphi_1(s) \, ds$$

$$\cdots\cdots\cdots$$

$$\varphi_n(x) = f(x)+\lambda \int_a^x K(x, s)\varphi_{n-1}(s) \, ds$$

$$\cdots\cdots\cdots$$

The sequence converges uniformly in the range $[a, b]$ for any values of $\lambda$. In point of fact, it is clear that

$$|\varphi_1(x)| \leqslant |f(x)|+|\lambda| \int_a^x |K(x, s)||\varphi_0(s)|\,ds$$

$$\leqslant B+|\lambda| AB_0(x-a), \quad \text{where} \quad |\varphi_0(s)| \leqslant B_0$$

$$|\varphi_2(x)| \leqslant |f(x)|+|\lambda| \int_a^x |K(x, s)||\varphi_1(s)|\,ds$$

$$\leqslant B+|\lambda|A \int_a^x \{B+|\lambda|AB_0(s-a)\}\,ds$$

$$= B+|\lambda| AB(x-a)+|\lambda|^2A^2B_0\frac{(x-a)^2}{2!}$$

In general,

$$|\varphi_n(x)| \leqslant B+|\lambda| AB(x-a)+\ldots+|\lambda^{n-1}| A^{n-1}B\frac{(x-a)^{n-1}}{(n-1)!}$$
$$+|\lambda|^nA^nB_0\frac{(x-a)^n}{n!}$$

Since the series $\sum_{n=1}^{\infty} B|\lambda|^nA^n(x-a)^n/n!$ converges uniformly in $[a, b]$, and its partial sums are the majorants for the functions $\varphi_n(x)$, it follows that the sequence $\{\varphi_n(x)\}$ will also converge uniformly; $\bar{\varphi}(x) = \lim_{n\to\infty} \varphi_n(x)$ is, clearly, the solution of (34) and, in fact, the unique solution.

## 8.5 APPROXIMATE METHODS OF SOLUTION OF FREDHOLM'S INTEGRAL EQUATIONS OF THE SECOND KIND

The method of successive approximations described in Section 8.4 can serve as an approximate method for the solution of integral equations. As the approximate solution we must take the functions $\varphi_n(x)$ given by (28)–(30).

A second method for solving integral equations involves the approximation of the kernel $K(x, s)$ by the degenerate kernel

$$\overline{K}(x, s) = \sum_{i=1}^{n} a_i(x)b_i(s)$$

The solution of the equation with the kernel $\bar{K}(x, s)$ will be the approximate solution of the original equation.

A third method consists of the following. The ranges $[a, b]$ of the variables $x$ and $s$ are divided into a grid of $n$ identical parts by points $x_i, s_j$, and the integral $\int_a^b K(x, s)\varphi(s)\,\mathrm{d}s$ in the integral equation is replaced by an integral sum. As a result,

$$\varphi(x) \cong \sum_{j=1}^{n} K(x, s_j)\varphi_j \nabla^2 s_j + f(x)$$

Assuming in this expression that $x$ is equal to $x_i (i = 1, 2, \ldots, n)$, let us consider the set of equations

$$\varphi_i = \lambda \sum_{j=1}^{n} K_{ij}\varphi_j \nabla^2 s_j + f_i \qquad (i = 1, 2, \ldots, n) \tag{35}$$

where

$$\varphi_i = \varphi(x_i), \qquad K_{ij} = K(x_i, s_j), \qquad f_i = f(x_i), \qquad \nabla^2 s_j = s_{j+1} - s_j$$

Solving this system for $\varphi_i$, we obtain the values of the approximate solution at the grid points. We shall not pause to consider these approximate methods in any detail and refer the reader to specialist literature.

## 8.6 FREDHOLM'S THEOREM

In this section and in the next chapter we shall confine our attention to Fredholm's integral equations of the second kind

$$\varphi(x) - \lambda \int_a^b K(x, s)\varphi(s)\,\mathrm{d}s = f(x) \tag{36}$$

**8.6.1** The homogeneous equation

$$\varphi(x) = \lambda \int_a^b K(x, s)\varphi(s)\,\mathrm{d}s \tag{37}$$

has the trivial solution $\varphi(x) \equiv 0$ for any value of the parameter $\lambda$. However, for some values of $\lambda$, it may also have non-trivial solutions.

*Definition* The values of $\lambda$, for which Equation (37) has non-trivial solutions (i.e. solutions which are not identically zero), are called

the eigenvalues of (37) [the kernel $K(x, s)$] and the corresponding solutions $\varphi(x)$ are called the eigenfunctions of the equation (kernel). We now have the following theorem.

*Theorem 1* If in Equation (36) the parameter $\lambda$ is not equal to the eigenvalue of the corresponding homogeneous Equation (37), then (36) can have only the unique solution. The existence of the solution will be established later.

*Proof* Let $\varphi_1(x)$ and $\varphi_2(x)$ be two solutions of (36). We then have

$$\varphi_1(x)-\lambda\int_a^b K(x, s)\varphi_1(s)\,\mathrm{d}s \equiv f(x)$$

$$\varphi_2(x)-\lambda\int_a^b K(x, s)\varphi_2(s)\,\mathrm{d}s \equiv f(x)$$

and, hence,

$$(\varphi_1-\varphi_2)-\lambda\int_a^b K(x, s)(\varphi_1-\varphi_2)\,\mathrm{d}s \equiv 0$$

Consequently, the difference $\varphi(x) = \varphi_1(x)-\varphi_2(x)$ is a solution of the homogeneous equation. Since $\lambda$ is not an eigenvalue, it follows that $\varphi(x) = \varphi_1(x)-\varphi_2(x) \equiv 0$. This proves the theorem.

**8.6.2** It will now be convenient to recall some theorems on systems of linear algebraic equations.

*Theorem A* A necessary and sufficient condition for the homogeneous system of equations

$$\sum_{j=1}^{n} a_{ij}x_j = 0 \qquad (i = 1, 2, \ldots, n) \tag{38}$$

to have only a trivial solution, i.e. a solution consisting only of zeros, is that the determinant of the system should be different from zero.

*Theorem B* If the determinant of the homogeneous system (38) is equal to zero, the system has $p = n-r$ linearly independent solutions where $r$ is the rank of the matrix of the system.

*Theorem C* If the homogeneous system of equations (38) has only a trivial solution, then the corresponding non-homogeneous system

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \qquad (i = 1, 2, \ldots, n) \tag{39}$$

has aunique solution for any values of the right-hand side, $b_i$.

**8.6.3** It was shown in Section 8.5 that the approximate solution of (36) can be obtained by replacing this equation with the corresponding system of linear algebraic equations

$$\varphi_m - \lambda \sum_{j=1}^{n} K_{jm} \varphi_j \nabla^2 s_j = f_m \qquad (m = 1, 2, \ldots, n) \tag{40}$$

and then solving this system.

Similarly, certain theorems for systems of linear algebraic equations can be extended to Fredholm's integral equations of the second kind. For integral equations these theorems are called Fredholm's theorems. We shall indicate below one of the methods of obtaining Fredholm's theorems without giving detailed proof.

The function $\varphi_n(x)$ which is equal to the solution of the system (40) at the corresponding grid points and is linear between them will be called a polygonal function corresponding to the solution of the system (40). We then have the following theorem.

*Theorem 2* A polygonal function $\varphi_n(x)$ corresponding to the solution of (40) tends uniformly to the solution of the integral equation (36) as $n \to \infty$.

We shall not give a proof of this theorem and will proceed to the method of deducing Fredholm's theorems.

Suppose that $\lambda$ is not an eigenvalue of the kernel $K(x, s)$. The homogeneous Equation (37) will then have only the trivial solution. Therefore, in view of Theorems A and 2, we conclude that the corresponding system of algebraic equations which replaces the integral Equation (37), i.e. the system

$$\tilde{\varphi}_m - \lambda \sum_{j=1}^{n} K_{jm} \tilde{\varphi}_j \nabla^2 s_j = 0 \qquad (m = 1, 2, \ldots, n)$$

has a non-zero determinant. Consequently, the system of equations

$$\varphi_m - \lambda \sum_{j=1}^{a} K_{jm} \varphi_j \nabla^2 s_j = f_m \qquad (m = 1, 2, \ldots, n)$$

which replaces the inhomogeneous integral Equation (36) has a unique solution. The polygonal function $\varphi_n(x)$ corresponding to this solution for $n \to \infty$ will, in accordance with Theorem 2, tend uniformly to the solution of (36). This means that the following theorem is valid.

*Fredholm's first theorem* For any $\lambda$ which is not an eigenvalue, Equation (36) has a unique solution.

*Remark* Since the determinants of the system (40) and of the transposed system

$$\psi_m - \lambda \sum_{j=1}^{n} K_{mj}\psi_j \nabla^2 s_j = f_m \qquad (m = 1, 2, \ldots, n)$$

coincide, it follows that for any $\lambda$ which is not an eigenvalue of the kernel $K(x, s)$, the adjoint integral equation

$$\psi(x) - \lambda \int_a^b K(s, x)\psi(s)\,\mathrm{d}s = f(x)$$

will also have a unique solution.

Let us consider now the case where $\lambda$ is equal to one of the eigenvalues. We then have the following theorem.

*Fredholm's second theorem* If $\lambda$ is an eigenvalue of the kernel $K(x, s)$ then both the homogeneous integral Equation (37) and its adjoint equation have a finite number of linearly independent solutions.

This theorem follows from the fact that the homogeneous system of algebraic equations corresponding to equation (37) has a finite number of linearly independent solutions, in accordance with Theorem B.

*Fredholm's third theorem* Let $\lambda$ be an eigenvalue of the kernel $K(x, s)$. In order that (36) have a solution, it is necessary and sufficient that the function $f(x)$ on the right-hand side of (36) be orthogonal to all the eigenfunctions of the adjoint homogeneous equation corresponding to this eigenvalue. The necessity of this condition is readily proved. In fact, if $\varphi(x)$ is a solution of (36), then

$$\varphi(x) - \lambda \int_a^b K(x, s)\varphi(s)\,\mathrm{d}s \equiv f(x)$$

Let us multiply this identity by the eigenfunction $\psi(x)$ of the adjoint equation and integrate the result (with respect to $x$) within the range $[a, b]$. We obtain

$$\int_a^b f(x)\psi(x)\,dx = \int_a^b \varphi(x)\psi(x)\,dx - \lambda\int_a^b \psi(x)\int_a^b K(x, s)\varphi(s)\,ds\,dx$$

Since

$$\lambda\int_a^b \psi(x)\int_a^b K(x, s)\varphi(s)\,ds\,dx = \int_a^b \varphi(s)\lambda\int_a^b K(x, s)\psi(x)\,dx\,ds$$

and

$$\lambda\int_a^b K(x, s)\psi(x)\,dx \equiv \psi(s)$$

it follows that

$$\int_a^b f(x)\psi(x)\,dx = \int_a^b \varphi(x)\psi(x)\,dx - \int_a^b \varphi(s)\psi(s)\,ds = 0$$

which was to be proved.

The sufficient condition is more difficult to establish. The proof can be given, for example, first for the corresponding system of algebraic equations; then, by going to the limit in the polygonal functions, one can extend this result to integral equations also. We shall not give this proof here.

Suppose that there are $r$ linearly independent eigenfunctions corresponding to a given eigenvalue $\lambda$. We then have the following theorem.

*Theorem* If in (36) $\lambda$ is one of the eigenvalues and the solution of (36) exists, i.e. $f(x)$ is orthogonal to the corresponding eigenfunctions of the adjoint equation, the solution of (36) will also be any function

$$\varphi(x) = \varphi_0(x) + \sum_{q=1}^{r} C_q\varphi_q(x)$$

where $\varphi_0(x)$ is a solution of (36), $\varphi_q(x)$ are eigenfunctions of the kernel $K(x, s)$ corresponding to the eigenvalue $\lambda$ and $C_q$ are arbitrary constants.

*Remark* In Section 8.4 it was shown that the inhomogeneous Volterra equation has a unique solution for any values of the parameter $\lambda$. Consequently, in accordance with Fredholm's theorems, the Volterra equation has no eigenvalues.

# 9

# Integral Equations with Symmetric Kernels

In this chapter we shall consider Fredholm's equations with symmetric kernels. The kernel $K(x, s)$ is called symmetric if, for all $x$ and $s$, the following identity is satisfied in the square $a \leqslant x, s \leqslant b$:

$$K(x, s) \equiv K(s, x)$$

If the kernel $K(x, s)$ is symmetric, it is evident that all the iterated kernels $K_n(x, s)$ are also symmetric. We recall that, for the sake of simplicity, we are confining our attention only to kernels continuous for $a \leqslant x, s \leqslant b$.

Equations with symmetric kernels are frequently encountered in mathematical physics. They have a number of specific properties, the chief of which is given by the following theorem.

*Theorem 1* Any continuous symmetric kernel, which is not identically equal to zero, has at least one eigenvalue.

We shall not prove the theorem here and will merely note that it is possible to find asymmetric kernels which do not possess eigenvalues. For example, the kernel

$$K(x, s) = \sin x \cos s, \quad 0 \leqslant x, \quad s \leqslant 2\pi$$

and all Volterra kernels are examples of this.

The set of all eigenvalues of a kernel (equation) is called the eigenvalue spectrum of the kernel (equation) or, simply, the spectrum of the kernel (equation).

## 9.1 SIMPLEST PROPERTIES OF EIGENFUNCTIONS AND EIGENVALUES

It is evident that the following properties must hold.

*Property 1* If $\varphi(x)$ is an eigenfunction corresponding to an eigenvalue $\lambda$, then $C\varphi(x)$, where $C$ is an arbitrary constant, is also an eigenfunction corresponding to the same eigenvalue $\lambda$.

The constant factor $C$ can be chosen so that the norm of the eigenfunction $C\varphi(x)$, i.e. $\|C\varphi\| = \sqrt{\int_a^b C^2\varphi^2(x)\,\mathrm{d}x}$, is equal to unity, so that $\|C\varphi\| = 1$. We shall assume henceforth that all eigenfunctions are normalised to unity in this way.

*Property 2* If two eigenfunctions $\varphi_1(x)$ and $\varphi_2(x)$ correspond to the same eigenvalue $\lambda$, then for all constants $C_1$ and $C_2$ the functions $C_1\varphi_1(x)+C_2\varphi_2(x)$ will also be functions corresponding to the same eigenvalue $\lambda$.

Let us now establish the following property.

*Property 3* The eigenfunctions $\varphi_1(x)$ and $\varphi_2(x)$ corresponding to different eigenvalues $\lambda_1$ and $\lambda_2$ are orthogonal in the range $[a, b]$, i.e.

$$\int_a^b \varphi_1(x)\varphi_2(x)\,\mathrm{d}x = 0$$

*Proof* By hypothesis we have

$$\frac{1}{\lambda_1}\varphi_1(x) \equiv \int_a^b K(x, s)\varphi_1(s)\,\mathrm{d}s, \qquad \frac{1}{\lambda_2}\varphi_2(x) \equiv \int_a^b K(x, s)\varphi_2(s)\,\mathrm{d}s$$

Let us multiply the first of these by $\varphi_2(x)$ and the second by $\varphi_1(x)$ and subtract one from the other. If we now integrate the resulting identity with respect to $x$ within the range $[a, b]$, we obtain

$$\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)\int_a^b \varphi_1(x)\varphi_2(x)\,\mathrm{d}x = \int_a^b\int_a^b K(x, s)\varphi_1(s)\varphi_2(x)\,\mathrm{d}s\,\mathrm{d}x$$

$$- \int_a^b\int_a^b K(x, s)\varphi_2(s)\varphi_1(x)\,\mathrm{d}s\,\mathrm{d}x$$

Changing the order of integration in the second term on the right, and bearing in mind the fact that the kernel is symmetric, we have

$$\int_a^b\int_a^b K(x,s)\varphi_2(s)\varphi_1(x)\,ds\,dx = \int_a^b\int_a^b K(x,s)\varphi_1(x)\varphi_2(s)\,dx\,ds$$

$$= \int_a^b\int_a^b K(x,s)\varphi_1(s)\varphi_2(x)\,ds\,dx$$

Consequently,

$$\left(\frac{1}{\lambda_1}-\frac{1}{\lambda_2}\right)\int_a^b \varphi_1(x)\varphi_2(x)\,dx = 0$$

The orthogonality property follows at once.

If the eigenfunctions corresponding to a given eigenvalue $\lambda$ are made orthogonal, any two linearly independent eigenfunctions $\varphi_1(x)$ and $\varphi_2(x)$ will be orthogonal.

We shall assume henceforth that the orthogonalisation procedure has been performed whenever necessary. Consequently, a set of eigenfunctions can be considered as orthonormal.

*Property 4* All eigenvalues of integral equations with symmetric kernels are real.

*Proof* Let us suppose that $\lambda = \alpha+i\beta$, $\beta \neq 0$ is a complex eigenvalue, and $\varphi(x) = \psi_1(x)+i\psi_2(x)$ is the corresponding eigenfunction. We then have

$$\psi_1(x)+i\psi_2(x) \equiv (\alpha+i\beta)\int_a^b K(x,s)[\psi_1(s)+i\psi_2(s)]\,ds$$

and hence

$$\psi_1(x) \equiv \alpha\int_a^b K(x,s)\psi_1(s)\,ds-\beta\int_a^b K(x,s)\psi_2(s)\,ds$$

$$\psi_2(x) \equiv \alpha\int_a^b K(x,s)\psi_2(s)\,ds+\beta\int_a^b K(x,s)\psi_1(s)\,ds$$

Let us multiply the second of these identities by $i$ and subtract the results from the first identity. We have

$$\psi_1(x)-i\psi_2(x) \equiv (\alpha-i\beta)\int_a^b K(x, s)[\psi_1(s)-i\psi_2(s)]\,\mathrm{d}s$$

Therefore

$$\bar{\lambda} = \alpha-i\beta \quad \text{and} \quad \bar{\varphi}(x) = \psi_1(x)-i\psi_2(x)$$

are, respectively, the eigenvalue and the corresponding eigenfunction. Since $\bar{\lambda} \neq \lambda$ ($\beta \neq 0$), it follows from Property 3 that the functions $\varphi(x)$ and $\bar{\varphi}(x)$ are orthogonal, i.e.

$$\int_a^b \varphi(x)\bar{\varphi}(x)\,\mathrm{d}x = \int_a^b \{\psi_1^2(x)+\psi_2^2(x)\}\,\mathrm{d}x = 0$$

Hence, since the functions $\psi_1(x)$ and $\psi_2(x)$ are continuous, it follows that $\psi_1(x) \equiv \psi_2(x) \equiv 0$. Hence $\varphi(x) \equiv 0$, which is impossible. This proves the property.

*Property 5* In each finite range $[A, B]$ there is only a finite number (including zero) of eigenvalues.

*Proof* Let us suppose that in a given range $[A_0, B_0]$ there is an infinite set of eigenvalues. Let us select out of this set a certain infinite sequence of eigenvalues $\{\tilde{\lambda}_n\}$. Suppose that $\{\bar{\varphi}_n(x)\}$ is the sequence of corresponding eigenfunctions. Since the family of functions $\{\bar{\varphi}_n(x)\}$ are orthonormal, the Fourier coefficients of the kernel $K(x, s)$ for this family are $(1/\tilde{\lambda}_n)\tilde{\varphi}_n(x)$. It follows that the Bessel inequality

$$\sum_{n=1}^{\infty} \frac{\tilde{\varphi}_n^2(x)}{\tilde{\lambda}_n^2} \leqslant \int_a^b K^2(x, s)\,\mathrm{d}s$$

must hold. Consequently, for any integer $p > 0$

$$\sum_{n=1}^{p} \frac{\tilde{\varphi}_n^2(x)}{\tilde{\lambda}_n^2} \leqslant \int_a^b K^2(x, s)\,\mathrm{d}s$$

Integrating this inequality with respect to $x$ within the range $[a, b]$, we obtain

$$\sum_{n=1}^{p} \frac{1}{\tilde{\lambda}_n^2} \leqslant \int_a^b\int_a^b K^2(x, s)\,\mathrm{d}s\,\mathrm{d}x$$

Since all the $\tilde{\lambda}_n$ lie in the finite range $[A_0, B_0]$, it follows that all the numbers $\tilde{\lambda}_n^2$ are less than $B^2$, $\tilde{\lambda}_n^2 < B^2$, where $B^2 = \max\{A_0^2, B_0^2\}$. If we replace all the $\tilde{\lambda}_n^2$ in the sum $\sum_{n=1}^{p} \frac{1}{\tilde{\lambda}_n^2}$ by the larger number $B^2$, we obtain for any integer $p$

$$\sum_{n=1}^{p} \frac{1}{B^2} \leqslant \int_a^b \int_a^b K^2(x, s)\,\mathrm{d}s\,\mathrm{d}x$$

which is impossible since the series $\sum_{n=1}^{\infty} \frac{1}{B^2}$ is divergent and consequently, for sufficiently large $p$, the sum $\sum_{n=1}^{p} \frac{1}{B^2}$ will be greater than $\int_a^b \int_a^b K^2(x, s)\,\mathrm{d}s\,\mathrm{d}x$.

From Property 5 it follows directly that (1) all the eigenvalues can be numbered in increasing order of their absolute magnitudes, i.e.

$$|\lambda_1| \leqslant |\lambda_2| \leqslant \dots \leqslant |\lambda_n| \leqslant \dots$$

and (2) if the eigenvalue spectrum is infinite, then $|\lambda_n| \to \infty$ as $n \to \infty$.

*Property 6* To each eigenvalue $\lambda$ there corresponds a finite number $q$ of eigenfunctions $\varphi_1(x), \varphi_2(x), \dots, \varphi_q(x)$.

*Proof* Let us suppose that to a given eigenvalue $\tilde{\lambda}$ there corresponds an infinite sequence of eigenfunctions $\tilde{\varphi}_1(x), \tilde{\varphi}_2(x), \dots, \tilde{\varphi}_n(x), \dots$

It follows from the Bessel inequality that for any integer $p > 0$ we have

$$\sum_{n=1}^{p} \frac{\tilde{\varphi}_n^2(x)}{\tilde{\lambda}^2} \leqslant \int_a^b K^2(x, s)\,\mathrm{d}s$$

Integrating this inequality with respect to $x$ within the range $[a, b]$, and bearing in mind that the eigenfunctions are normalised, we have for any integer $p > 0$

$$\sum_{n=1}^{p} \frac{1}{\tilde{\lambda}^2} \leqslant \int_a^b \int_a^b K^2(x, s)\,\mathrm{d}s\,\mathrm{d}x \quad \text{for} \quad p \leqslant \tilde{\lambda}^2 \int_a^b \int_a^b K^2(x, s)\,\mathrm{d}s\,\mathrm{d}x$$

which is impossible. It follows that to each eigenvalue $\lambda$ there corresponds only a finite number of eigenfunctions.

It follows from Section 8.3 that a degenerate symmetric kernel can only have a finite spectrum. In fact, in order that a homogeneous system of linear equations ((27) for $\beta_i = 0$; Chapter 8) for the coefficients $C_i$ should have a non-zero solution, it is necessary that the determinant of this system $D(\lambda)$ be zero. The eigenvalues can be found from this equation. It is evident that it has only a finite number of roots. The converse is also true; if a symmetric kernel $K(x, s)$ has a finite spectrum, it is degenerate. In fact, suppose that $\lambda_1, \lambda_2, \ldots, \lambda_n$ is the spectrum of the kernel and $\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x)$ is the set of all the corresponding eigenfunctions of the kernel (complete system). Consider the symmetric continuous function

$$K^{(n)}(x, s) = K(x, s) - \sum_{p=1}^{n} \frac{1}{\lambda_p} \varphi_p(x)\varphi_p(s)$$

If it is not identically equal to zero, then by Theorem 1 it has at least one eigenvalue $\mu$ and the corresponding function is

$$\psi(x) \equiv \mu \int_a^b K^{(n)}(x, s)\psi(s)\,ds$$

The function $\psi(x)$ is orthogonal to all the eigenfunctions $\varphi_q(x)$ of the kernel $K(x, s)$ since

$$\int_a^b \psi(x)\varphi_q(x)\,dx = \mu \int_a^b \varphi_q(x) \int_a^b K^{(n)}(x, s)\psi(s)\,ds\,dx$$

$$= \mu \int_a^b \int_a^b K^{(n)}(x, s)\varphi_q(x)\psi(s)\,ds\,dx$$

On changing the order of integration we obtain

$$\int_a^b \int_a^b K^{(n)}(x, s)\psi(s)\varphi_q(x)\,ds\,dx = \int_a^b \psi(s) \int_a^b \left\{K(x, s) - \sum_{p=1}^{n} \frac{\varphi_p(x)\varphi_p(s)}{\lambda_p}\right\}\varphi_q(x)\,dx\,ds$$

Since the functions $\varphi_p(x)$ are orthonormal, the last integral is equal to

$$\int_a^b \psi(s)\left\{\int_a^b K(x,s)\varphi_q(x)\,\mathrm{d}x - \frac{\varphi_q(s)}{\lambda_q}\right\}\mathrm{d}s = \int_a^b \psi(s)\left\{\frac{1}{\lambda_q}\varphi_q(s) - \frac{1}{\lambda_q}\varphi_q(s)\right\}\mathrm{d}s = 0$$

when $\mu$ and $\varphi(x)$ are the eigenvalue and eigenfunction of the kernel $K(x, s)$, since

$$\mu\int_a^b K(x,s)\psi(s)\,\mathrm{d}s = \mu\int_a^b\left\{K^{(n)}(x,s) + \sum_{p=1}^{n}\frac{\varphi_p(x)\varphi_p(s)}{\lambda_p}\right\}\psi(s)\,\mathrm{d}s$$

$$= \mu\int_a^b K^{(n)}(x,s)\psi(s)\,\mathrm{d}s = \psi(x)$$

In deriving this expression we have used the orthogonality of $\psi(x)$ and $\varphi_p(x)$. Since $\psi(x)$ is the eigenfunction of the kernel $K(x, s)$ and the functions $\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x)$ form a complete system of eigenfunctions of the kernel $K(x, s)$, it follows that $\psi(x)$ should be a linear combination of the functions $\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x)$. This, however, is impossible since $\psi(x)$ is orthogonal to all these functions. We cannot, therefore, assume that $K^{(n)}(x, s) \not\equiv 0$. Consequently, $K^{(n)}(x, s) \equiv 0$ or

$$K(x,s) = \sum_{p=1}^{n}\frac{1}{\lambda_p}\varphi_p(x)\varphi_p(s)$$

i.e. the kernel $K(x, s)$ is degenerate. This means that the following theorem must be valid.

*Theorem 2* In order that the spectrum of a symmetric kernel be finite, it is necessary and sufficient that the kernel be degenerate.

## 9.2 THE SPECTRUM OF ITERATED KERNELS

Let us substitute

$$A\varphi \equiv \int_a^b K(x,s)\varphi(s)\,\mathrm{d}s$$

It follows from the definition of iterated kernels that

$$A(A\varphi) = A^2\varphi = \int_a^b K_2(x,s)\varphi(s)\,ds$$

and, in general,

$$A^n\varphi = A(A^{n-1}\varphi) = \int_a^b K_n(x,s)\varphi(s)\,ds$$

For eigenfunctions $\varphi_p(x)$ and eigenvalues $\lambda_p$ we have

$$\begin{aligned}\varphi_p(x) &= \lambda_p A\varphi_p = \lambda_p A(\lambda_p A\varphi_p)\\ &= \lambda_p^2 A^2\varphi_p = \dots = \lambda_p^n A^n\varphi_p = \lambda_p^n \int_a^b K_n(x,s)\varphi_p(s)\,ds\end{aligned}$$

from which it follows that the following theorem must be valid.

*Theorem 3* If $\varphi_p(x)$ and $\lambda_p$ are the eigenfunctions and eigenvalues of the kernel $K(x, s)$, respectively, then $\varphi_p(x)$ and $\lambda_p^n$ will be the eigenfunction and eigenvalue of the kernel $K_n(x, s)$.

The following theorem is also valid.

*Theorem 4* If $\mu$ is an eigenvalue of the kernel $K_n(x, s)$, then at least one of the (real) roots of the $n$-th power of $\mu$ will be an eigenvalue of the kernel $K(x, s)$.

To prove this result, it will be convenient to introduce the following lemma.

*Lemma* If $h_1, h_2, \dots, h_n$ are the roots of the equation $h^n = \mu$, then

$$h_1^s + h_2^s + \dots + h_n^s = 0$$

for $s = 1, 2, \dots, n-1$.

*Proof* It is well known that $h_m = \sqrt[n]{\mu}\,\xi^m$ where $\sqrt[n]{\mu}$ is a root of the equation $h^n = \mu$, $\xi = e^{i2\pi/n}$. We then have

$$\begin{aligned}h_1^s + h_2^s + \dots + h_n^s &= \sqrt[n]{\mu^s}\,(1+\xi^s+\xi^{2s}+\dots+\xi^{s(n-1)})\xi^s\\ &= \xi^s \sqrt[n]{\mu^s}\,\frac{\xi^{sn}-1}{\xi^s-1} = 0\end{aligned}$$

since $\xi^{sn} = 1$.

*Proof of the theorem* Let $\psi(x)$ be an eigenfunction of the kernel $K_n(x, s)$ corresponding to the eigenvalue $\mu$. We shall determine the functions $\varphi_p(x)$ from the formula

$$\varphi_p(x) = \frac{1}{n}(\psi + h_p A\psi + h_p^2 A^2\psi + \ldots + h_p^{n-1} A^{n-1}\psi) \tag{1}$$

Summing these equations over $p$ between $p = 1$ and $p = n$, and bearing in mind the above lemma, we obtain

$$\psi(x) \equiv \sum_{p=1}^{n} \varphi_p(x)$$

It follows from this identity that among functions $\varphi_p(x)$ there is at least one which is not identically equal to zero. It is readily seen that $\varphi_p(x) \equiv h_p A\varphi_p$. In point of fact, if we apply the operator $A$ to the identity given by (1) and multiply the result by $h_p$, we obtain

$$h_p A\varphi_p \equiv \frac{1}{n}(h_p A\psi + h_p^2 A^2\psi + \ldots + h_p^{n-1} A^{n-1}\psi) + \frac{1}{n} h_p^n A^n \psi$$

or

$$h_p A\varphi_p \equiv \varphi_p(x) - \frac{1}{n}\psi(x) + \frac{1}{n} h_p^n A^n \psi \equiv \varphi_p(x)$$

since $h_p^n = \mu$ and $\mu A^n \psi \equiv \psi$.

It follows that functions $\varphi_p(x)$ which are not identically equal to zero are the eigenfunctions of the kernel $K(x, s)$ and $h_p$ are the corresponding eigenvalues. From Property 4, the kernel $K(x, s)$ has only real eigenvalues. Consequently, the functions $\varphi_p(x)$ corresponding to complex roots $h_p$ are identically zero. When $n$ is odd, there is only one real root $\sqrt[n]{\mu} = h_p$ which should, in fact, be the eigenvalue of the kernel $K(x, s)$ and $\varphi_p(x) \equiv \psi(x)$ is its eigenfunction. If $n$ is odd, there are two real roots. Suppose that $\varphi_1(x)$ and $\varphi_2(x)$ are the corresponding eigenfunctions. We then have

$$\psi(x) \equiv \varphi_1(x) + \varphi_2(x) \tag{2}$$

It follows that when $n$ is odd, each of the eigenfunctions of the kernel $K(x, s)$ will also be an eigenfunction of the kernel $K(x, s)$. For even $n$ each eigenfunction of $K(x, s)$ will either coincide with the eigenfunction of the kernel $K(x, s)$ (one of the functions $\varphi_1(x)$ or $\varphi_2(x)$ in (2) may be identically zero) or will be a combination of the eigenfunctions of $K(x, s)$. This means that if $\{\lambda_p\}$ and $\{\varphi_p(x)\}$

represent the sets of all the eigenvalues and all the eigenfunctions of the kernel $K(x, s)$, then $\{\lambda_p^n\}$ and $\{\varphi_p(x)\}$ are the sets of all the eigenvalues and eigenfunctions of $K_n(x, s)$.

## 9.3 EXPANSION OF ITERATED KERNELS

In this section we shall show that for any $n \geqslant 3$, the following expansion is valid

$$K_n(x, s) = \sum_{p=1}^{\infty} \frac{\varphi_p(x)\varphi_p(s)}{\lambda_p^n} \tag{3}$$

in which the series converges absolutely and uniformly within the range $a \leqslant x,\ s \leqslant b$.

We shall show, to begin with, that the series on the right of (3) converges absolutely and uniformly for $a \leqslant x, s \leqslant b$. For this let us estimate the finite series

$$\sum_{p=m}^{m+q} \frac{1}{|\lambda_p^n|} |\varphi_p(x)\varphi_p(s)| \leqslant \frac{1}{2|\lambda_m^{n-2}|} \sum_{p=m}^{m+q} \left[\frac{\varphi_p^2(x)}{\lambda_p^2} + \frac{\varphi_p^2(s)}{\lambda_p^2}\right] \tag{4}$$

We have used the inequality

$$|A \cdot B| \leqslant \frac{1}{2}(A^2+B^2)$$

and the fact that $|\lambda_p|$ tends monotonically to zero as $p \to \infty$. From Bessel's inequality

$$\sum_{p=1}^{\infty} \frac{\varphi_p^2(x)}{\lambda_p^2} \leqslant \int_a^b K^2(x, s)\,\mathrm{d}s \leqslant D$$

where $D =$ const and $D > 0$. Therefore, for $q > 0$

$$\sum_{p=m}^{m+q} \left|\frac{\varphi_p(x)\varphi_p(s)}{\lambda_p^n}\right| \leqslant \frac{D}{|\lambda_m^{n-2}|} \tag{5}$$

Since $|\lambda_m| \to \infty$ as $m \to \infty$, it follows from (5) in accordance with Cauchy's criterion that the series (3) converge absolutely and uniformly.

Let

$$\Phi(x, s) = \sum_{p=1}^{\infty} \frac{1}{\lambda_p^n} \varphi_p(x)\varphi_p(s)$$

This function is continuous within the range $a \leqslant s$, $s \leqslant b$. We must show that $K_n(x, s) \equiv \Phi(x, s)$. Let us suppose that this is not so. The symmetric function

$$Q(x, s) = K_n(x, s) - \Phi(x, s)$$

will by Theorem 1 have an eigenvalue $\mu$ and an eigenfunction $\psi(x)$, i.e.

$$\psi(x) \equiv \mu \int_a^b Q(x, s)\psi(s)\,\mathrm{d}s$$

The function $\psi(x)$ is orthogonal to all the eigenfunctions $\varphi_r(x)$ of the kernel $K(x, s)$ since

$$\int_a^b \psi(x)\varphi_r(x)\,\mathrm{d}x = \mu \int_a^b\int_a^b Q(x, s)\psi(s)\varphi_r(x)\,\mathrm{d}s\,\mathrm{d}x$$

$$= \mu \int_a^b \psi(s) \int_a^b \Bigg\{K_n(x, s)$$

$$- \sum_{p=1}^{\infty} \frac{1}{\lambda_p^n}\varphi_p(x)\varphi_p(s)\Bigg\}\varphi_r(x)\,\mathrm{d}x\,\mathrm{d}s$$

$$= \mu \int_a^b \psi(x)\left\{\int_a^b K_n(x, s)\varphi_r(x)\,\mathrm{d}x - \frac{\varphi_r(s)}{\lambda_r^n}\right\}\mathrm{d}s = 0$$

and because $\varphi_r(s) \equiv \lambda_r^n \int_a^b K_n(x, s)\varphi_r(x)\,\mathrm{d}x$. The function $\psi(x)$ is an eigenfunction of the kernel $K_n(x, s)$ since

$$\psi(x) \equiv \mu \int_a^b Q(x, s)\psi(s)\,\mathrm{d}s$$

$$\equiv \mu \int_a^b \left\{K_n(x, s) - \sum_{p=1}^{\infty} \frac{\varphi_p(x)\varphi_p(s)}{\lambda_p^n}\right\}\psi(s)\,\mathrm{d}s$$

$$\equiv \mu \int_a^b K_n(x, s)\psi(s)\,\mathrm{d}s$$

We have used the orthogonality of the function $\psi(x)$ to all the eigenfunctions $\varphi_p(x)$. Consequently, as was shown in Section 9.2, $\psi(x)$ should be a linear combination of the functions $\varphi_p(x)$. However, this is impossible since $\psi(x)$ is orthogonal to all the functions $\varphi_p(x)$. Therefore, it cannot be assumed that $Q(x, s) \not\equiv 0$.

*Remark* The expansion given by (3) is valid also for $K_2(x, s)$ ($n = 2$) and, under certain additional conditions, for $K(x, s)$. We shall not prove this here.

## 9.4 HILBERT–SCHMIDT THEOREM

We shall now prove one of the fundamental theorems of the theory of linear integral equations which has many applications. This is the expansion theorem.

*Hilbert–Schmidt theorem* If the function $\varphi(x)$ can be represented by

$$f(x) = \int_a^b K(x, s) h(s)\, \mathrm{d}s \tag{6}$$

where $h(s)$ is piecewise-continuous in $[a, b]$, it can be represented by an expansion in terms of the eigenfunctions of the kernel $K(x, s)$, i.e.

$$f(x) = \sum_{p=1}^{\infty} f_p \varphi_p(x) \tag{7}$$

where

$$f_p = \int_a^b f(x) \varphi_p(x)\, \mathrm{d}x$$

This series converges absolutely and uniformly in the range $[a, b]$.

To prove this theorem the following lemma will be convenient.

*Lemma* In order that a continuous function $Q(x)$ be orthogonal to the kernel $K(x, s)$, i.e.

$$\int_a^b K(x, s) Q(s)\, \mathrm{d}s \equiv 0 \tag{8}$$

it is necessary and sufficient that it should be orthogonal to each eigenfunction of the kernel, i.e.

$$\int_a^b Q(x)\varphi_p(x)\,\mathrm{d}x = 0 \qquad (p = 1, 2, \ldots) \tag{9}$$

*Proof* Sufficiency:

$$\int_a^b Q(x)\varphi_p(x)\,\mathrm{d}x = \lambda_p \int_a^b \int_a^b K(x, s)\varphi_p(s)Q(x)\,\mathrm{d}s\,\mathrm{d}x$$

$$= \lambda_p \int_a^b \varphi_p(s)\Big\{\int_a^b K(x, s)Q(x)\,\mathrm{d}x\Big\}\,\mathrm{d}s = 0$$

since the inner integral is zero.

Necessity: consider the auxiliary integral

$$J_1 = \int_a^b \int_a^b K_4(x, s)Q(x)Q(s)\,\mathrm{d}s\,\mathrm{d}x$$

It is zero since, using Equation (3) for $n = 4$ and Equation (9), we obtain

$$J_1 = \int_a^b \int_a^b \sum_{p=1}^{\infty} \frac{\varphi_p(x)\varphi_p(s)}{\lambda_p^4} Q(x)Q(s)\,\mathrm{d}s\,\mathrm{d}x$$

$$= \sum_{p=1}^{\infty} \frac{1}{\lambda_p^4} \int_a^b \varphi_p(x)Q(x)\,\mathrm{d}x \int_a^b \varphi_p(s)Q(s)\,\mathrm{d}s = 0$$

Since $K_4(x, s) = \int_a^b K_2(x, t)K_2(t, s)\,\mathrm{d}t$, it follows that

$$0 = J_1 = \int_a^b \int_a^b \Big\{\int_a^b K_2(x, t)K_2(t, s)\,\mathrm{d}t\Big\} Q(x)Q(s)\,\mathrm{d}s$$

$$= \int_a^b \Big\{\Big[\int_a^b K_2(x, t)Q(x)\,\mathrm{d}x\Big]\Big[\int_a^b K_2(t, s)Q(s)\,\mathrm{d}s\Big]\Big\}\,\mathrm{d}t$$

$$= \int_a^b \Big\{\int_a^b K_2(x, t)Q(x)\,\mathrm{d}x\Big\}^2 \mathrm{d}t$$

Consequently,

$$\int_a^b K_2(x, t)Q(x)\,\mathrm{d}x \equiv 0 \tag{10}$$

where we have used the symmetry of the kernel $K_2(x, s)$. Multiplying (10) by $Q(t)$ and integrating the result with respect to $t$ within the range $[a, b]$, we obtain

$$\int_a^b\int_a^b K_2(x, t)Q(x)Q(t)\,\mathrm{d}x\,\mathrm{d}t = 0$$

If we replace $K_2(x, t)$ in this equation by the integral $\int_a^b K(x, \xi)K(\xi, t)\,\mathrm{d}\xi$ and perform transformations similar to those carried out above, we obtain

$$\int_a^b K(x, \xi)Q(x)\,\mathrm{d}x \equiv 0$$

This proves the lemma.

*Remark* The Fourier coefficients $f_p$ of the function $f(x)$ are equal to $h_p/\lambda_p$, where $h_p$ are the Fourier coefficients of $h(s)$. In fact,

$$f_p = \int_a^b f(x)\varphi_p(x)\,\mathrm{d}x = \int_a^b \varphi_p(x)\int_a^b K(x, s)h(s)\,\mathrm{d}s\,\mathrm{d}x$$

$$= \int_a^b h(s)\int_a^b K(x, s)\varphi_p(x)\,\mathrm{d}x = \int_a^b h(s)\frac{\varphi_p(s)}{\lambda_p}\,\mathrm{d}s = \frac{h_p}{\lambda_p}$$

and, therefore, instead of the series given by (7) we may consider the series

$$f(x) = \sum_{p=1}^{\infty} \frac{h_p}{\lambda_p}\varphi_p(x) \tag{11}$$

*Proof of the theorem* We shall first show that (11) converges absolutely and uniformly. From the Cauchy inequality we have

$$\sum_{p=n}^{n+q}\left|\frac{h_p}{\lambda_p}\varphi_p(x)\right| \leqslant \sqrt{\sum_{p=n}^{n+q} h_p^2}\,\sqrt{\sum_{p=n}^{n+q}\frac{\varphi_p^2(x)}{\lambda_p^2}} \tag{12}$$

From Bessel's inequality we have

$$\sum_{p=1}^{\infty} h_p^2 \leqslant \int_a^b h^2(s)\,\mathrm{d}s \quad \text{and} \quad \sum_{p=1}^{\infty} \frac{\varphi_p^2(x)}{\lambda_p^2} \leqslant \int_a^b K^2(x, s)\,\mathrm{d}s \leqslant D$$

Consequently, the series $\sum_{p=1}^{\infty} h_p^2$ converges and therefore the sum $\sum_{p=n}^{n+q} h_p^2$ can be made less than $\varepsilon/D$, where $\varepsilon$ is an arbitrary number, if $n$ is taken to be sufficiently large. Hence, for large enough $n$,

$$\sum_{p=n}^{n+q} \left| \frac{h_p}{\lambda_p} \varphi_p(x) \right| < \varepsilon \quad \text{when} \quad x \in [a, b]$$

which means that (11) converges absolutely and uniformly. Let

$$Q(x) = \sum_{=1}^{\infty} \frac{h_p}{\lambda_p} \varphi_p(x) - f(x)$$

The function $Q(x)$ is continuous in $[a, b]$. It is orthogonal to all the functions $\varphi_p(x)$. In fact,

$$\int_a^b f(x)\varphi_r(x)\,\mathrm{d}x = \int_a^b \left\{ \sum_{p=1}^{\infty} \frac{h_p}{\lambda_p} \varphi_p(x) - f(x) \right\} \varphi_r(x)\,\mathrm{d}x$$

$$= \frac{h_r}{\lambda_r} - f_r = 0$$

Consequently, in accordance with the above lemma, it is orthogonal to the kernel $K(x, s)$, i.e.

$$\int_a^b K(x, s) Q(x)\,\mathrm{d}x \equiv 0 \tag{13}$$

Next, since the functions $Q(x)$ and $\varphi_p(x)$ are orthogonal,

$$\int_a^b Q^2(x)\,\mathrm{d}x = \int_a^b Q(x) \left\{ \sum_{p=1}^{\infty} \frac{h_p}{\lambda_p} \varphi_p(x) - f(x) \right\} \mathrm{d}x$$

$$= -\int_a^b Q(x) f(x)\,\mathrm{d}x$$

Replacing $f(x)$ in accordance with (6), and using (13), we obtain

$$\int_a^b Q^2(x)\,\mathrm{d}x = -\int_a^b Q(x)\int_a^b K(x, s)h(s)\,\mathrm{d}s\,\mathrm{d}x$$

$$= -\int_a^b h(s)\int_a^b K(x, s)Q(x)\,\mathrm{d}x\,\mathrm{d}s = 0$$

Consequently,

$$Q(x) = \sum_{p=1}^{\infty} \frac{h_p}{\lambda_p}\,\varphi_p(x) - f(x) \equiv 0$$

and this proves the theorem.

## 9.5 EXPANSION OF THE SOLUTION OF THE INHOMOGENEOUS EQUATION

Suppose that in the equation

$$\varphi(x) = \lambda\int_a^b K(x, s)\varphi(s)\,\mathrm{d}s + f(x) \tag{14}$$

$\lambda$ is not equal to any of the eigenvalues. Then by Fredholm's first theorem this equation has the unique solution

$$\varphi(x) = f(x) + \lambda g(x) \tag{15}$$

where

$$g(x) = \int_a^b K(x, s)\varphi(s)\,\mathrm{d}s$$

By the Hilbert–Schmidt theorem the function $g(x)$ can be represented by an expansion in terms of the eigenfunctions of the kernel $K(x, s)$:

$$g(x) = \sum_{p=1}^{\infty} C_p\varphi_p(x) \tag{16}$$

Substituting (15) into (14), we obtain

$$f(x) + \lambda\sum_{p=1}^{\infty} C_p\varphi_p(x) \equiv f(x) + \lambda\int_a^b K(x, s)\left\{f(s) + \lambda\sum_{p=1}^{\infty} C_p\varphi_p(s)\right\}\mathrm{d}s$$

or

$$\sum_{p=1}^{\infty} C_p \varphi_p(x) \equiv \int_a^b K(x, s) f(s)\, ds + \lambda \sum_{p=1}^{\infty} C_p \int_a^b K(x, s) \varphi_p(s)\, ds$$

We now apply the Hilbert–Schmidt theorem to the function

$$\int_a^b K(x, s) f(s)\, ds$$

and replace $\int_a^b K(x, s)\varphi_p(s)\, ds$ by $\varphi_p(x)/\lambda_p$. We obtain

$$\sum_{p=1}^{\infty} C_p \varphi_p(x) \equiv \sum_{p=1}^{\infty} \frac{f_p}{\lambda_p} \varphi_p(x) + \lambda \sum_{p=1}^{\infty} C_p \frac{\varphi_p(x)}{\lambda_p}$$

and hence

$$C_p = \frac{f_p}{\lambda_p} + \frac{\lambda}{\lambda_p} C_p \quad \text{for} \quad C_p = \frac{f_p}{\lambda_p - \lambda}$$

It follows that the required solution of (14) can be represented by the absolutely and uniformly convergent series

$$\varphi(x) = f(x) + \lambda \sum_{p=1}^{\infty} \frac{f_p}{\lambda_p - \lambda} \varphi_p(x) \tag{17}$$

If $\lambda$ is equal to an eigenvalue $\lambda_r$, which corresponds to eigenfunctions $\varphi_r(x)$, $\varphi_{r+1}(x)$, ..., $\varphi_{r+q}(x)$, then $\lambda_r = \lambda_{r+1} = \ldots = \lambda_{r+q}$.

In this case, it is evident from the formulae for the coefficients $C_p$ that

$$f_r = f_{r+1} = \ldots = f_{r+q} = 0$$

or

$$\int_a^b f(x) \varphi_{r+t}(x)\, dx = 0 \qquad (t = 0, 1, 2, \ldots, q)$$

i.e. the function $f(x)$ should be orthogonal to all the eigenfunctions of the kernel corresponding to the eigenvalue $\lambda_r$. The coefficients

$C_r, C_{r+1}, \ldots, C_{r+q}$ are then undefined (they remain arbitrary) and the solution of (14) can be written in the form

$$\varphi(x) = C_r\varphi_r(x)+C_{r+1}\varphi_{r+1}(x)+\ldots+C_{r+q}\varphi_{r+q}(x)$$

$$+\lambda_r \sum_p{}' \frac{f_p}{\lambda_p-\lambda_r}\varphi_p(x) \tag{18}$$

where $\Sigma'$ represents summation with respect to all values of $p$ except for $p = r, r+1, \ldots, r+q$.

*Remark* An equation with an antisymmetric kernel of the form

$$\varphi(x) = \lambda\int_a^b K(x, s)\rho(s)\varphi(s)\,\mathrm{d}s$$

where $\rho(s)$ is an unknown function, $\rho(s) \geqslant 0$ in $[a, b]$, and $K(x, s)$ is a symmetric function, can be reduced to an equation with a symmetric kernel with respect to the function $\psi(x) = \varphi(x)\sqrt{\rho(x)}$:

$$\psi(x) = \lambda\int_a^b K(x, s)\sqrt{\rho(x)\rho(s)}\,\psi(s)\,\mathrm{d}s$$

## 9.6 EXPANSION THEOREM

It was shown in Section 8.2 that the boundary-value problem

$$L[\Phi]+\lambda\rho\Phi \equiv \frac{\mathrm{d}}{\mathrm{d}x}[k\Phi']-q\Phi+\lambda\rho\Phi = 0 \tag{19}$$

$$\alpha_1\Phi'(0)-\beta_1\Phi(0) = 0, \qquad \alpha_2\Phi'(l)+\beta_2\Phi(l) = 0 \tag{20}$$

is equivalent to the integral equation

$$\Phi(x) = \lambda\int_a^b G(x, s)\rho(s)\Phi(s)\mathrm{d}s \tag{21}$$

or

$$\Psi(x) = \lambda\int_a^b K_1(x, s)\Psi(s)\mathrm{d}s \tag{22}$$

where $K_1(x, s) = G(x, s)\sqrt{\rho(x)\rho(s)}$, $\Psi(x) = \Phi(x)\sqrt{\rho(x)}$ and $G(x, s)$ is Green's function for the boundary-value problem (19)–(20).

Consequently, the eigenvalue and eigenfunction of the boundary-value problem (19)–(20) coincide with the eigenvalue and eigenfunction of the kernel $K_1(x, s)$. This fact enables us to derive the expansion theorem from the Hilbert–Schmidt theorem. In fact, let $f(x)$ be a function belonging to class $A$ (see Section 4.1), so that

$$L[f] \equiv \frac{\mathrm{d}}{\mathrm{d}x}[kf'] - qf = -F(x)$$

will be a piecewise-continuous function, and by Hilbert's first theorem (see Section 8.2)

$$f(x) = \int G(x, s) F(s) \,\mathrm{d}s$$

Consequently, by the Hilbert–Schmidt theorem $f(x)$ can be represented by an absolutely and uniformly convergent series in terms of the eigenfunctions $\{\Phi_p(x)\}$ of the boundary-value problem (19)–(20)

$$f(x) = \sum_{p=1}^{\infty} C_p \Phi_p(x)$$

This proves the expansion theorem for the one-dimensional case (Section 4.1).

## 9.7 CLASSIFICATION OF KERNELS

Let us now consider one further application of the Hilbert–Schmidt theorem. Positive definite (negative definite) kernels are a particularly interesting class of symmetric kernels. The kernel $K(x, s)$ is called positive definite (negative definite) if for any piecewise-continuous function $h(x,)$ the integral form

$$J = \int_a^b \int_a^b K(x, s) h(x) h(s) \,\mathrm{d}s \,\mathrm{d}x \tag{23}$$

is positive (negative).

It is readily shown that the necessary and sufficient condition for the kernel $K(x, s)$ to be positive (negative) definite is that all its eigenvalues $\lambda_p$ be positive (negative).

In fact, the function

$$f(x) = \int_a^b K(x, s)h(s)\,\mathrm{d}s$$

can by the Hilbert–Schmidt theorem be represented by the uniformly convergent (in the range $[a, b]$) series

$$f(x) = \int_a^b K(x, s)h(s)\,\mathrm{d}s = \sum_{p=1}^{\infty} \frac{h_p}{\lambda_p}\varphi_p(x) \tag{24}$$

Multiplying both sides of this equation by $h(x)$ and integrating with respect to $x$ in the range $[a, b]$, we obtain

$$J = \int_a^b\int_a^b K(x, s)h(s)h(x)\,\mathrm{d}s\,\mathrm{d}x = \sum_{p=1}^{\infty} \frac{h_p^2}{\lambda_p} \tag{25}$$

Consequently, if all the eigenvalues $\lambda_p$ are positive (negative) the form given by (23) is positive (negative). If the form (23) is positive for any piecewise-continuous function $h(x)$, then for $h(x) = \varphi_n(x)$ the formula given by (25) yields

$$\int_a^b\int_a^b K(x, s)\varphi_n(s)\varphi_n(x)\,\mathrm{d}s\,\mathrm{d}x = \frac{1}{\lambda_n}$$

Consequently, $\lambda_n > 0$ and, similarly, for the negative form. For positive definite (negative definite) kernels we have the following theorem.

*Theorem* If the kernel $K(x, s)$ is positive (negative) definite and continuous with respect to the variables $x, s$ in the range $a \leqslant x$, $s \leqslant b$, then it can be represented by the uniformly convergent series

$$K(x, s) = \sum_{p=1}^{\infty} \frac{\varphi_p(x)\varphi_p(s)}{\lambda_p}$$

where $\varphi_p$ and $\lambda_p$ are the eigenfunctions and eigenvalues of the kernel. We shall not prove this theorem here.

It is important to note that all the theorems and facts referring to the Fredholm equations and described in this chapter and in

Section 8.5 are valid also for the multidimensional case including kernels of the form

$$K(P, Q) = \frac{H(P, Q)}{|PQ|^{\alpha}}, \qquad \alpha < \frac{d}{2}$$

where $H(P, Q)$ is a continuous kernel, $|PQ|$ is the distance between the points $P$ and $Q$ and $d$ is the dimensionality of the space.

## 9.8 THE SPECTRUM OF SYMMETRIC KERNELS SPECIFIED IN AN INFINITE DOMAIN

We have considered integral equations with an infinite range of integration $[a, b]$. For integral equations with an infinite range of integration, the above results will not, in general, be valid. Thus, for symmetric kernels specified in a finite domain we can establish the following facts.

1. The spectrum is discrete.
2. The spectrum of a non-degenerate kernel is infinite.
3. To each eigenvalue there corresponds a finite number of linearly independent eigenfunctions.

For symmetric kernels specified in an infinite domain these results will not, in general, be valid. This is indicated by the following example.

*Example 1*

$$\varphi(x) = \lambda \int_0^{\infty} \sin(xs)\varphi(s)\,\mathrm{d}s \tag{26}$$

has only two eigenvalues, namely $\lambda_1 = \sqrt{2/\pi}$, $\lambda_2 = -\sqrt{2/\pi}$, and to each of them there corresponds an infinite set of linearly independent eigenfunctions.

To prove this we shall use the following well-known results:

$$\sqrt{\frac{\pi}{2}} \int_0^{\infty} \sin(xs)\mathrm{e}^{-as}\,\mathrm{d}s = \frac{x\sqrt{\frac{\pi}{2}}}{a^2+x^2}$$

$$\int_0^{\infty} \frac{s\sin(xs)}{a^2+s^2}\,\mathrm{d}s = \frac{\pi}{2}\mathrm{e}^{-ax}$$

where $x > 0$ and $a > 0$. Adding and subtracting these equations, we obtain

$$\int_0^\infty \sin(xs)\left[\sqrt{\frac{\pi}{2}}\,e^{-as}+\frac{s}{a^2+s^2}\right]ds = \sqrt{\frac{\pi}{2}}\left[\sqrt{\frac{\pi}{2}}\,e^{-ax}+\frac{x}{a^2+x^2}\right]$$

and

$$\int_0^\infty \sin(xs)\left[\sqrt{\frac{\pi}{2}}\,e^{-as}-\frac{s}{a^2+s^2}\right]ds = \sqrt{\frac{\pi}{2}}\left[\sqrt{\frac{\pi}{2}}\,e^{-ax}-\frac{x}{a^2+x^2}\right]$$

Therefore, $\lambda_1 = \sqrt{2/\pi}$ and $\lambda_2 = -\sqrt{2/\pi}$ are the eigenvalues of Equation (26) and

$$\varphi_1(x) = \sqrt{\frac{\pi}{2}}\,e^{-ax}+\frac{x}{a^2+x^2} \quad \text{and} \quad \varphi_2(x) = \sqrt{\frac{\pi}{2}}\,e^{-ax}-\frac{x}{a^2+x^2}$$

are the corresponding eigenfunctions for any $a$. The functions $\varphi_1(x)$ [and $\varphi_2(x)$] corresponding to different values of the parameter $a$ are linearly independent. Consequently, to each of the eigenvalues $\lambda_1$, $\lambda_2$ there corresponds an infinite set of linearly independent eigenfunctions. We shall now show that the equation given by (26) has no other eigenvalues. To establish this let us substitute $\varphi(s)$ defined by this equation into the right-hand side of (26). We obtain

$$\varphi(x) = \lambda^2 \int_0^\infty \sin(xs) \int_0^\infty \sin(st)\varphi(t)\,dt\,ds$$

Comparing this with the Fourier integral

$$\varphi(x) = \frac{2}{\pi}\int_0^\infty \sin(xs) \int_0^\infty \sin(st)\varphi(t)\,dt\,ds$$

we find that $\lambda^2 = 2/\pi$.

*Example 2* Consider equations of the form

$$\varphi(x) = \lambda \int_{-\infty}^{\infty} H(|x-s|)\varphi(s)\,ds \tag{27}$$

where $H(z)$ has the following properties.

1. It is continuous and positive for all $z \geqslant 0$.
2. There exists a positive number $A$ ($A \leqslant \infty$) such that the integral $\int_0^\infty H(z)\cosh\alpha z\,\mathrm{d}z$ converges for all positive $A < \alpha$ and diverges for $\alpha = A$.

All the eigenvalues and all the eigenfunctions of this equation can be found. We shall seek the solution in the form $\varphi(x) = \mathrm{e}^{\alpha x}$. Substituting this function into (27) we obtain

$$\mathrm{e}^{\alpha x} = \lambda \int_{-\infty}^{x} H(x-s)\mathrm{e}^{\alpha s}\mathrm{d}s + \lambda \int_{x}^{\infty} H(s-x)\mathrm{e}^{\alpha s}\mathrm{d}s$$

or, by changing the integration variable ($x-s = z$ in the first integral and $s-x = z$ in the second), we have

$$\mathrm{e}^{\alpha x} = \lambda \int_0^\infty H(z)\mathrm{e}^{\alpha(x-z)}\mathrm{d}z + \lambda \int_0^\infty H(z)\mathrm{e}^{\alpha(x+z)}\mathrm{d}z$$

and hence

$$1 = 2\lambda \int_0^\infty H(z)\cosh\alpha z\,\mathrm{d}z$$

Therefore, for

$$\lambda = \lambda(\alpha) = \frac{1}{2\int_0^\infty H(z)\cosh\alpha z\,\mathrm{d}z} \tag{28}$$

the function $\varphi(x) = \mathrm{e}^{\alpha x}$, where $\alpha < A$, will be a solution of Equation (27), i.e. its eigenfunction corresponds to the eigenvalue $\lambda = \lambda(\alpha)$.

Similarly, we find that for $\alpha < A$ the function $\mathrm{e}^{-\alpha x}$ will also be an eigenfunction of Equation (27) corresponding to the same eigenvalue $\lambda = \lambda(\alpha) = \lambda(-\alpha)$.

Since $\cosh\alpha z$ increases monotonically with $\alpha$, it follows that for $\lambda(\alpha)$ defined by (28), the values of $\lambda(0) = 1/(2\int_0^\infty H(z)\,\mathrm{d}z)$ will increase monotonically and continuously in the range $0 \leqslant \alpha < A$ up to the value $\lambda(A) = 0$. Therefore, to each value $\lambda \in [0, \lambda(0)]$ there corresponds a definite value $\alpha$ ($\alpha \geqslant 0$), which can be determined from (28) and, consequently, the solutions of (27) are

$$C_1\mathrm{e}^{\alpha x} + C_2\mathrm{e}^{-\alpha x}$$

where $C_1$ and $C_2$ are arbitrary constants. The solution corresponding to $\lambda(0)$ is

$$\varphi(x) = \lim_{\alpha\to 0} \frac{e^{\alpha x} - e^{-\alpha x}}{2\alpha} = x$$

which is readily verified by direct substitution.

When $\lambda > 1/(2\int\limits_0^\infty H(z)\,dz)$, the solution must be sought in the form $e^{\pm i\beta x}$. Substitution of these functions into (27) then shows that the values $\lambda = \lambda(i\beta) = \lambda(-i\beta)$, where

$$\lambda(i\beta) = \lambda(-i\beta) = \frac{1}{2\int\limits_0^\infty H(z)\cos\alpha z\,dz} > \frac{1}{2\int\limits_0^\infty H(z)\,dz}$$

correspond to the real solutions $\cos\beta x$ and $\sin\beta x$ of Equation (27).

Therefore, the spectrum of Equation (27) is continuous; all the non-negative $\lambda$ are its eigenvalues. Thus, if we take the equation

$$\varphi(x) = \lambda \int\limits_{-\infty}^{\infty} e^{-|x-s|}\varphi(s)\,ds \tag{29}$$

then $H(z) = e^{-z}$, $\lambda(\alpha) = (1-\alpha^2)/2$, $\lambda(0) = 1/2$ and $A = 1$.

# PART TWO

The methods described in Part 1 (for example, the method of separation of variables in cylindrical and spherical polar coordinates), lead to the so-called special functions, i.e. cylindrical, spherical, and other functions. These functions are solutions of equations with singular points of the form

$$\frac{d}{dx}\left[k(x)\frac{dy}{dx}\right]-q(x)y=0$$

where $k(x)$ vanishes at one or more points in the range of the variable $x$. The properties and applications of cylindrical and spherical functions as well as certain special polynomials will be discussed in Part 2.

# 10

# Gamma Function

The gamma function (or the Euler integral of the second kind) is defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \tag{1}$$

This function has the following properties.

*Property 1* $\Gamma(z)$ is bounded and continuous for $\operatorname{Re} z > 0$.

*Proof* Consider a closed domain $D \equiv \{0 < \delta \leqslant \operatorname{Re} z \leqslant N\}$, where $\delta$ and $N$ are arbitrary fixed numbers. For all $z \in \bar{D}$, we have

$$|t^{z-1}e^{-t}| \leqslant t^{\delta-1} \qquad 0 \leqslant t \leqslant 1$$
$$|t^{z-1}e^{-t}| \leqslant e^{-t} t^{N-1} \qquad 1 \leqslant t < \infty$$

Consequently, the function

$$f(t) = \begin{cases} t^{\delta-1}, & 0 \leqslant t \leqslant 1 \\ t^{N-1}e^{-t} & 1 < t < \infty \end{cases}$$

is the majorant for $|t^{z-1}e^{-t}|$ in the range $0 \leqslant t < \infty$ for all $z \in \bar{D}$. Since the integral $\int_0^\infty f(t) dt = \int_0^1 t^{\delta-1} dt + \int_1^\infty t^{N-1}e^{-t} dt$ converges, it follows that the integral $\int_0^\infty t^{z-1} e^{-t} dt$ converges uniformly for $z \in \bar{D}$. It follows that $\Gamma(z)$ is bounded and continuous in the domain $\bar{D}$ and, hence, in $0 < \operatorname{Re} z < \infty$, since $\delta$ and $N$ are arbitrary.

*Property 2* $\Gamma(z)$ is analytic for $\operatorname{Re} z > 0$. To prove this property, it will be sufficient to show that the integral $\int_C \Gamma(z)\,dz$ evaluated over an arbitrary piecewise-smooth closed contour $C$ lying in the domain $\overline{D}$ is zero. By Morera's theorem, $\Gamma(z)$ will then be analytic in the domain $D$ and, consequently, for $\operatorname{Re} z > 0$,

$$\int_C \Gamma(z)\,dz = \int_C \left(\int_0^\infty t^{z-1}e^{-t}\,dt\right)dz = \int_0^\infty e^{-t}\left(\int_C t^{z-1}\,dz\right)dt = 0$$

since by Cauchy's integral theorem

$$\int_C t^{z-1}\,dz = 0$$

The change in the order of integration is justified since the integral $\int_0^\infty t^{z-1}e^{-t}\,dt$ converges uniformly for $z \in \overline{D}$.

*Property 3* The identity

$$\Gamma(z+1) \equiv z\Gamma(z) \tag{2}$$

is satisfied for $\operatorname{Re} z > 0$. The validity of this property can be established directly by integrating by parts:

$$\Gamma(z+1) = \int_0^\infty t^z e^{-t}\,dt = -e^{-t}t^z\Big|_0^\infty + z\int_0^\infty t^{z-1}e^{-t}\,dt$$

$$= z\int_0^\infty t^{z-1}e^{-t}\,dt = z\Gamma(z)$$

Successive application of (2) yields

$$\Gamma(z) \equiv \frac{\Gamma(z+n+1)}{z(z+1)\dots(z+n-1)(z+n)} \tag{3}$$

*Property 4* The function $\Gamma(z)$ can be continued analytically to the entire $z$ plane with the aid of (3) except for the points $z = 0, -1, -2, \dots, -n, \dots$, at which $\Gamma(z)$ has first-order poles with residues given by

$$\operatorname{Re} \Gamma(-n) = \frac{(-1)^n}{n!}$$

*Property 5* $\Gamma(n+1) = n!$
Direct evaluation yields $\Gamma(1) = 1$. Using (3) we have

$$\Gamma(n+1) = n! \tag{2$_1$}$$

Therefore, $\Gamma(z)$ can be regarded as a generalisation of the factorial function to arbitrary complex numbers.

*Property 6* The following relationship is valid:

$$\Gamma(z)\,\Gamma(1-z) \equiv \frac{\pi}{\sin \pi z} \tag{4}$$

*Remark* It is sufficient to establish the validity of Property 6 for $z = x$, where $0 < x < 1$. In view of the uniqueness of analytic functions this property will then be valid for all $z$.

*Proof*

$$\Gamma(x) = \int_0^\infty t^{x-1}\mathrm{e}^{-t}\mathrm{d}t = 2\int_0^\infty u^{2x-1}\mathrm{e}^{-u^2}\mathrm{d}u \qquad (t = u^2)$$

$$\Gamma(1-x) = \int_0^\infty t^{-x}\mathrm{e}^{-t}\mathrm{d}t = 2\int_0^\infty v^{-2x+1}\mathrm{e}^{-v^2}\mathrm{d}v \qquad (t = v^2)$$

and, consequently,

$$\Gamma(x)\Gamma(1-x) = 4\int_0^\infty u^{2x-1}\mathrm{e}^{-u^2}\mathrm{d}u\int_0^\infty v^{-2x+1}\mathrm{e}^{-v^2}\mathrm{d}v$$

$$= 4\int_0^\infty\int_0^\infty \mathrm{e}^{-(u^2+v^2)}\left(\frac{u}{v}\right)^{2x-1}\mathrm{d}u\,\mathrm{d}v$$

Substituting $u = r\cos\varphi$ and $v = r\sin\varphi$, we obtain

$$\Gamma(x)\Gamma(1-x) = 4\int_0^\infty\int_0^{\pi/2} \mathrm{e}^{-r^2}r(\cot\varphi)^{2x-1}\mathrm{d}r\,\mathrm{d}\varphi$$

$$= 4\int_0^\infty \mathrm{e}^{-r^2}r\,\mathrm{d}r\int_0^{\pi/2}(\cot\varphi)^{2x-1}\mathrm{d}\varphi = 2\int_0^{\pi/2}(\cot\varphi)^{2x-1}\mathrm{d}\varphi$$

If in the last integral we substitute $\cot^2\varphi = e^\beta$ we obtain

$$\Gamma(x)\Gamma(1-x) = \int_{-\infty}^{\infty} \frac{e^{x\beta}\,d\beta}{1+e^\beta} = \frac{\pi}{\sin \pi x}$$

which was to be proved.

*Property 7* The function $\Gamma(z)$ has no zeros. Suppose that $z_0$ is a zero of the gamma function. It is clear that $z_0$ will not be equal either to a negative integer or to zero. From (4) we have

$$\lim_{z\to z_0} \Gamma(1-z) = \lim_{z\to z_0} \frac{\pi}{\Gamma(z)\sin \pi z} = \infty$$

Therefore, $z_0$ is a singularity of $\Gamma(1-z)$. However, by Property 4 the singular points of the gamma function are only positive integers. Consequently, $1-z_0 = -n$, where $n$ is an integer and $n \geqslant 0$, while $z_0 = 1+n$. It follows that

$$\Gamma(z_0) = \Gamma(n+1) = n! \neq 0$$

Hence, the supposed existence of the zero $z_0$ leads to a contradiction.

*Property 8*

$$\Gamma(z) = \frac{1}{e^{2\pi i z}-1}\int_\gamma t^{z-1}e^{-t}\,dt \tag{5}$$

where $\gamma$ is the contour shown in Fig. 10.1.

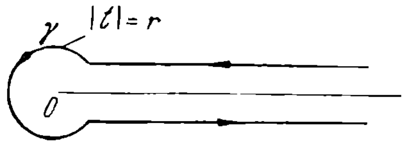



*Fig. 10.1*

To prove the validity of (5), let us establish the lemma

$$\int_{\gamma_1} t^{z-1}e^{-t}dt = \int_{\gamma_2} t^{z-1}e^{-t}dt$$

where $\gamma_1$ and $\gamma_2$ are the contours shown in Fig. 10.2.

Consider the integral evaluated over the contour $C$ shown in Fig. 10.3. This contour encloses a singly-connected domain in which

$e^{-t}t^{z-1}$ is an analytic function. Consequently, by Cauchy's integral theorem

$$\int_C e^{-t}t^{z-1}\,dt = 0$$

Moreover,

$$0 = \int_C e^{-t}t^{z-1}\,dt$$

$$= \int_{\gamma_2'} e^{-t}t^{z-1}\,dt + \int_{\gamma_1'} e^{-t}t^{z-1}\,dt + \int_{t_1t_2} e^{-t}t^{z-1}\,dt + \int_{t_3t_4} e^{-t}t^{z-1}\,dt$$

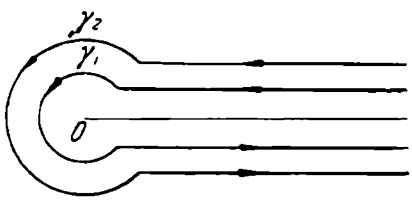



*Fig. 10.2*

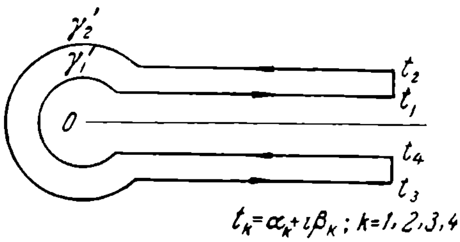



*Fig. 10.3*

where $\gamma_1'$, $\gamma_2'$ are shown in Fig. 10.3, and $t_k = \alpha_k + i\beta_k$ ($k = 1, 2, 3, 4$). Let us proceed to the limit in this equation as $\alpha \to \infty$, keeping $\beta_k$ constant. The integrals $\int_{\gamma_2'}$ and $\int_{\gamma_1'}$ will tend to $\int_{\gamma_2}$ and $-\int_{\gamma_1}$, respectively. If we can show that the integrals $\int_{t_1t_2}$ and $\int_{t_3t_4}$, evaluated over the ranges $t_1t_2$ and $t_3t_4$, will then tend to zero, this will prove the lemma. Consider $\int_{t_1t_2}$, bearing in mind that $z = x+iy$:

$$\left|\int_{t_1t_2} e^{-t}t^{z-1}\,dt\right| \leqslant \int_{t_1t_2} |e^{-t}t^{z-1}||dt| = e^{-\alpha}\int_{t_1t_2} |t^{z-1}||dt|$$

$$= e^{-\alpha}\int_{t_1t_2} |t|^{x-1}e^{-y\arg t}\,dt$$

Since $|t| < 2\alpha$ and $\arg t \leqslant 2\pi$, we have

$$e^{-\alpha}\int_{t_1t_2} |t|^{x-1}e^{-y\arg t}|dt| < e^{-\alpha}(2\alpha)^{x-1}e^{2\pi|y|}|t_2 - t_1|$$

For any fixed $z$ the last product tends to zero as $\alpha \to \infty$. Therefore, $\int\limits_{t_1 t_2} e^{-t} t^{z-1} dt \to 0$ as $\alpha \to \infty$. The fact that the integral over $t_3 t_4$ tends to zero can be established in a similar way. This proves the lemma.

Using the above lemma, we can take the contour $\gamma$ in

$$F(z) = \int\limits_{\gamma} e^{-t} t^{z-1} dt$$

as consisting of a circle $\gamma_0$ of radius $r$ centred on the point $t = 0$ and the upper and lower edges of the cut along the real axis between $r$ and $\infty$:

$$F(z) = \int\limits_{\gamma_0} e^{-t} t^{z-1} dt - \int\limits_{\substack{r \\ \text{(upper edge)}}}^{\infty} e^{-t} t^{z-1} dt + \int\limits_{\substack{r \\ \text{(lower edge)}}}^{\infty} e^{-t} t^{z-1} dt$$

The integral $F_0(z) = \int\limits_{\gamma_0} e^{-t} t^{z-1} dt$ is an analytic function of $z$ everywhere. In fact, $F_0(z)$ is continuous in the entire plane and the integral $\int\limits_C F_0(z) dz = \int\limits_{\gamma_0} e^{-t} \int\limits_C t^{z-1} dz\, dt$ evaluated over any closed piecewise smooth contour $C$ is zero. Therefore, by Morera's theorem $F_0(z)$ is analytic everywhere.

The integral

$$F_1(z) = \int\limits_{\substack{r \\ \text{(upper edge)}}}^{\infty} e^{-t} t^{z-1} dt$$

can be written in the form of the sum of two integrals:

$$F_1(z) = \int\limits_r^1 + \int\limits_1^{\infty}$$

The integral $\int\limits_r^1 e^{-t} t^{z-1} dt$ is not an improper integral but is a continuous function of $z$. The integral $\int\limits_1^{\infty} e^{-t} t^{z-1} dt$ converges uniformly in any band $-N \leqslant \operatorname{Re} z \leqslant N$, since for all $t > 1$ we have $|e^{-t} t^{z-1}| \leqslant e^{-t} t^{N-1}$, and the integral $\int\limits_1^{\infty} e^{-t} t^{N-1} dt$ converges. Consequently, the integral $\int\limits_1^{\infty} e^{-t} t^{z-1} dt$ is also a continuous function of $z$ in any strip for which

$-N \leqslant \operatorname{Re} z \leqslant N$, and hence in the entire plane of the variable $z$. Hence, it follows that the function $F_1(z)$ is continuous in the entire plane of the variable $z$.

Moreover, using Morera's theorem, we can establish the analyticity of the function $F_1(z)$. The integral $\int_C F_1(z)\mathrm{d}z$, evaluated over an arbitrary piecewise-smooth closed contour $C$ is zero since $t^{z-1}$ is an analytic function of $z$ and

$$\int_C F_1(z)\mathrm{d}z = \int_C \left\{\int_r^1 + \int_1^\infty\right\}\mathrm{d}z$$

$$= \int_r^1 \left(\int_C \mathrm{e}^{-t}t^{z-1}\mathrm{d}z\right)\mathrm{d}t + \int_1^\infty \left(\int_C \mathrm{e}^{-t}t^{z-1}\mathrm{d}z\right)\mathrm{d}t = 0$$

The change in the order of integration is justified because the integral $\int_1^\infty \mathrm{e}^{-t}t^{z-1}\mathrm{d}t$ converges uniformly in any strip $-N \leqslant \operatorname{Re} z \leqslant N$. Finally,

$$F_2(z) = \underset{\text{(lower edge)}}{\int_r^\infty} \mathrm{e}^{-t}t^{z-1}\mathrm{d}t = \mathrm{e}^{2\pi i z} \underset{\text{(upper edge)}}{\int_r^\infty} \mathrm{e}^{-t}t^{z-1}\mathrm{d}t = \mathrm{e}^{2\pi i z}F_1(z)$$

Therefore, $F(z) = F_0(z) + (\mathrm{e}^{2\pi i z} - 1)F_1(z)$ is an analytic function of $z$ everywhere. Therefore, to prove Equation (5) it will be sufficient to establish this expression for $z = x > 0$ (see the remark in connection with Property 6).

*Proof* We have

$$F(x) = F_0(x) + (\mathrm{e}^{2\pi i x} - 1)F_1(x) \tag{6}$$

Let $\gamma_0$ contract to a point. $F_1(x)$ will then tend to $\Gamma(x)$ and $F_0(x)$ will tend to zero since

$$|F_0(x)| \leqslant \int_{\gamma_0} |\mathrm{e}^{-t}t^{x-1}|\,|\mathrm{d}t| \leqslant \int_0^{2\pi} \mathrm{e}^{-r}r^x\mathrm{d}\varphi \leqslant r^x\mathrm{e}^r 2\pi \underset{r\to 0}{\to} 0$$

$$(t = r\,\mathrm{e}^{i\varphi} \quad \text{on} \quad \gamma_0)$$

Consequently, by passing to the limit in (6) as $r \to 0$, we obtain Equation (5).

*Property 9* From Properties 6 and 8 we have

$$\frac{1}{\Gamma(z+1)}=\frac{e^{i\pi z}}{2\pi i}\int_{\gamma} e^{-t}t^{-z-1}\,dt \tag{7}$$

In fact

$$\frac{1}{\Gamma(z+1)}=\frac{\sin(\pi z+\pi)}{\pi}\,\Gamma(-z)=\frac{-\sin\pi z}{\pi}\,\Gamma(-z)$$

$$=\frac{e^{-i\pi z}-e^{i\pi z}}{2\pi i\,(e^{-2\pi i z}-1)}\int_{\gamma} e^{-t}t^{-z-1}\,dt=\frac{e^{i\pi z}}{2\pi i}\int_{\gamma} e^{-t}t^{-z-1}\,dt$$

The gamma function is plotted in Fig. 10.4.

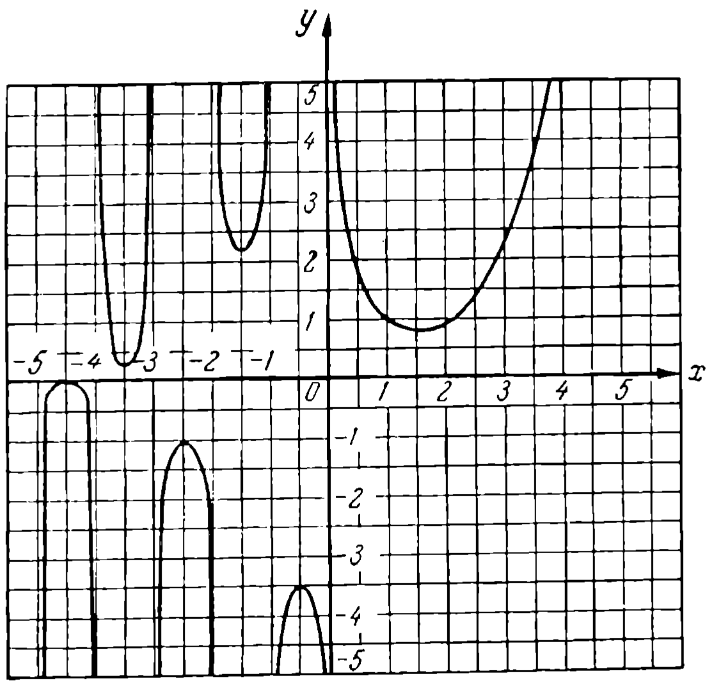



*Fig. 10.4*

# 11

# Cylindrical Functions

Many problems require the solution of differential equations of the form given by (1). This equation has to be solved in the case of problems considered in Chapter 1 by the method of separation of variables if the problems are expressed in terms of cylindrical (or polar) coordinates (vibrations of a circular membrane, cooling of a circular cylinder, and so on). The solutions of this equation will be investigated in this chapter.

## 11.1 BESSEL FUNCTIONS

**11.1.1** Consider the equation

$$L[w] \equiv z^2w''+zw'+(z^2-\nu^2)w = 0 \tag{1}$$

Solutions of this equation which are not identically zero are called cylindrical functions. We shall construct one class of cylindrical functions as follows. We shall seek the solution of (1) in the form of the series

$$w = z^\sigma(a_0+a_1z+a_2z^2+\ldots) \tag{2}$$

where $a_0 \neq 0$. We then have

$$zw' = z^\sigma[a_0\sigma+a_1(\sigma+1)z+a_2(\sigma+2)z^2+\ldots]$$
$$z^2w'' = z^\sigma[a_0\sigma(\sigma-1)+a_1(\sigma+1)\sigma z+a_2(\sigma+2)(\sigma+1)z^2+\ldots]$$

Substituting these expressions into (1) and collecting terms with the same powers of $z$, we obtain

$$z^{\sigma}[a_0\sigma^2 - a_0\nu^2] + z^{\sigma+1}[a_1(\sigma+1)^2 - a_1\nu^2] + z^{\sigma+2}[a_2(\sigma+2)^2 - a_2\nu^2 + a_0] + \dots + z^{\sigma+n}[a_n(\sigma+n)^2 - a_n\nu^2 + a_{n-2}] + \dots \equiv 0$$

If (2) is to be a solution of (1), we must have

$$\begin{aligned} a_0(\sigma^2-\nu^2) &= 0 \\ a_1[(\sigma+1)^2-\nu^2] &= 0 \\ a_2[(\sigma+2)^2-\nu^2]+a_0 &= 0 \\ \dots,\ a_n[(\sigma+n^2)-\nu^2]+a_{n-2} &= 0, \dots \end{aligned}$$

From the first equation we find that

$$\sigma = \pm\nu$$

since $a_0 \neq 0$. Let us take $\sigma = \nu$. From the second equation, we then have $a_1 = 0$. Moreover,

$$a_n = \frac{-a_{n-2}}{(\sigma+n)^2-\nu^2}, \qquad n = 2, 3, \dots$$

Since $\sigma = \nu$, it follows that

$$a_n = \frac{-a_{n-2}}{(2\nu+n)n}$$

and it is clear that

$$a_{2k+1} = 0$$

for all non-negative integers $k$ and

$$a_{2k} = \frac{-a_{2k-2}}{2^2(\nu+k)k} = \frac{(-1)^k a_0}{2^{2k}(\nu+k)(\nu+k-1)\dots(\nu+1)k!}$$

Substituting $a_0 = 1/2^{\nu}\Gamma(\nu+1)$ and using Equations (2) and $(2_1)$ of the preceding chapter, we obtain

$$a_{2k} = \frac{(-1)^k}{2^{2k+\nu}\Gamma(k+\nu+1)\Gamma(k+1)}$$

We therefore obtain the formal solution of Equation (1) in the form of the generalised power series

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k+\nu}}{\Gamma(k+\nu+1)\Gamma(k+1)} \tag{3}$$

This series is, in fact, the solution of (1) in the region in which it converges. It is readily verified using, for example, the d'Alembert test, that the series converges everywhere except possibly for $z = 0$.

Consequently, the function $J_\nu(z)$ is a solution of (1) everywhere except, possibly, for $z = 0$. This function is called Bessel's function of order $\nu$ (sometimes Bessel's function of the first kind). Since (1) is unaffected when $\nu$ is replaced by $-\nu$, the function $J_{-\nu}(z)$ is also a solution of (1). When $\nu$ is not an integer, the functions $J_\nu(z)$ and $J_{-\nu}(z)$ are linearly independent because one of them behaves as $z^\nu$ in the neighbourhood of $z = 0$ and the other behaves as $z^{-\nu}$. On the other hand, if $\nu$ is an integer (say, $\nu = n$), then

$$J_{-n}(z) \equiv (-1)^n J_n(z)$$

Let us prove this. We have

$$J_{-n}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{2k-n}}{\Gamma(k-n+1)\Gamma(k+1)} = \sum_{k=n}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{2k-n}}{\Gamma(k-n+1)\Gamma(k+1)}$$

since $\Gamma(k-n+1) = \infty$ for all $k = 0, 1, 2, \ldots, n-1$.

Substitute $k = s+n$ in the last sum. This yields

$$J_{-n}(z) = \sum_{s=0}^{\infty} \frac{(-1)^{n+s} \left(\frac{z}{2}\right)^{2s+n}}{\Gamma(s+1)\Gamma(s+n+1)} = (-1)^n J_n(z)$$

**11.1.2** The behaviour of Bessel's functions $J_\nu(z)$ and $J_{-\nu}(z)$ for non-integral $\nu$ in the neighbourhood of $z = 0$ is a manifestation of a property exhibited by linearly independent solutions of the following class of equations (with singular points)

$$\frac{d}{dx}\left[k(x)\frac{dy}{dx}\right] - q(x)y = 0 \tag{4}$$

in which $k(x) = (x-a)\varphi(x)$, $\varphi(a) \neq 0$ and $\varphi(x)$ is differentiable at $x = a$ and in the neighbourhood of this point. This property will be frequently used below. Let us formulate it more precisely.

*Theorem* If the solution $y_1(x)$ of Equation (4) is bounded in the neighbourhood of $x = a$, then any other solution $y_2(x)$ of (4), which is linearly independent of $y_1(x)$, is not bounded in the neighbourhood of $x = a$. Moreover, if $y_1(x) = (x = a)^m u_1(x)$ and $u_1(a) \neq 0$, then $y_2(x)$ has a singularity of the form $(x-a)^{-m}$ at the point $x = a$, and if $y_1(a) \neq 0$, then $y_2(x)$ has a logarithmic singularity at the point $x = a$.

*Proof* From the properties of the Wronskian, we have

$$y_2'(x)y_1(x)-y_2(x)y_1'(x) \equiv \frac{C}{k(x)}, \qquad C \neq 0$$

and hence

$$\frac{d}{dx}\left[\frac{y_2}{y_1}\right] \equiv \frac{C}{k(x)y_1^2(x)}$$

Integrating this identity within the range $[x, x_1]$, $a < x < x_1$, we obtain

$$y_2(x) = y_1(x)\left[\frac{y_2(x_1)}{y_1(x_1)} - \int\limits_x^{x_1} \frac{C\,dt}{k(t)y_1^2(t)}\right]$$

where $x_1$ is a fixed number such that the functions $\varphi(x)$ and $u_1(x)$ do not vanish in the range $[a, x_1]$.

Consider the first case: $y_1(x) = (x-a)^m u_1(x)$. We then have

$$y_2(x) = y_1(x)\left[\frac{y_2(x_1)}{y_1(x_1)} - C\int\limits_x^{x_1} \frac{dt}{(t-a)^{2m+1}\varphi(t)\,u_1^2(t)}\right]$$

Applying the mean value theorem to this integral, we obtain

$$y_2(x) = \frac{y_2(x_1)}{y_1(x_1)}\,y_1(x) - \frac{Cy_1(x)}{\varphi(\xi_1)\,u_1^2(\xi_1)}\int\limits_x^{x_1}\frac{dt}{(t-a)^{2m+1}}$$

or

$$y_2(x) = \frac{y_2(x_1)}{y_1(x_1)}\,y_1(x) + \frac{Cu_1(x)\,(x-a)^m}{2m\varphi(\xi_1)u_1^2(\xi_1)}\,(t-a)^{-2m}\Big|_x^{x_1}$$

$$x \leqslant \xi_1 \leqslant x_1, \quad \xi_1 = \xi_1(x)$$

and, therefore,

$$y_2(x) = A_1(x)\,y_1(x) + B_1(x)\,(x-a)^{-m}$$

where

$$A_1(x) = \frac{y_2(x_1)}{y_1(x_1)} + \frac{C}{2m(x_1-a)^{2m}\varphi(\xi_1)u_1^2(\xi)}$$

$$B_1(x) = \frac{-Cu_1(x)}{2m\varphi(\xi_1)\,u_1^2(\xi_1)}$$

Since $A_1(x)$ and $B_1(x)$ have no singularities at $x = a$, this proves the theorem for the first case. In the second case, similar calculations lead to the following result

$$y_2(x) = A_2(x)y_1(x) + B_2(x)\ln(x-a)$$

where

$$A_2(x) = \frac{y_2(x_1)}{y_1(x_1)} - \frac{C\ln(x_1-a)}{\varphi(\xi_2)u_1^2(\xi_2)}, \quad B_2(x) = \frac{C}{\varphi(\xi_2)u_1^2(\xi_2)}$$

$$x \leqslant \xi_2 \leqslant x_1, \quad \xi_2 = \xi_2(x)$$

Since the functions $A_2(x)$ and $B_2(x)$ have no singularities at $x = a$ this proves the theorem for the second case.

The above theorem is of considerable importance in formulating boundary-value problems for Equation (4) within the range $[a, b]$ where either one or both end-points of this range are singularities of the equation. If it is required to find the solution $y_1(x)$ which is bounded in $[a, b]$, then by writing the general solution in the form

$$y = C_1y_1(x) + C_2y_2(x)$$

we can find one of the arbitrary constants from the condition that the solution must be bounded ($C_2 = 0$). Therefore, the condition that the solution must be bounded plays the role of a boundary condition, so that it must be formulated as one of the boundary conditions.

**11.1.3** The Bessel functions $J_\nu(\lambda x)$ are orthogonal with the weight $\rho(x) = x$. More precisely, for any $\nu > -1$

$$\int_0^l xJ_\nu\left(\frac{\alpha}{l}x\right)J_\nu\left(\frac{\beta}{l}x\right)\mathrm{d}x = 0 \text{ when } \alpha \neq \beta \tag{5}$$

where both $\alpha$ and $\beta$ are the roots of one of the three equations

$$J_\nu(\gamma) = 0, \quad J_\nu'(\gamma) = 0, \quad \gamma J_\nu'(\gamma) + hJ_\nu(\gamma) = 0$$

*Remark* The equation

$$x^2y'' + xy' + (\lambda^2x^2 - \nu^2)y = 0 \tag{6}$$

can be reduced to (1) by substituting $\lambda x = z$. Consequently, the Bessel function $J_\nu(\lambda x)$ is a solution of (6).

*Proof of the orthogonality property* Consider the two identities

$$x\frac{d^2}{dx^2}J_\nu(\lambda x)+\frac{d}{dx}J_\nu(\lambda x)+\left(\lambda^2x-\frac{\nu^2}{x}\right)J_\nu(\lambda x)\equiv 0$$

$$x\frac{d^2}{dx^2}J_\nu(\mu x)+\frac{d}{dx}J_\nu(\mu x)+\left(\mu^2x-\frac{\nu^2}{x}\right)J_\nu(\mu x)\equiv 0$$

Let us multiply the first of these by $J_\nu(\mu x)$ and the second by $J_\nu(\lambda x)$ and then integrate with respect to $x$ in the range $[0, l]$ term by term. We have

$$\int_0^l x\left\{J_\nu(\mu x)\frac{d^2}{dx^2}J_\nu(\lambda x)-J_\nu(\lambda x)\frac{d^2}{dx^2}J_\nu(\mu x)\right\}dx$$

$$+\int_0^l\left\{J_\nu(\mu x)\frac{d}{dx}J_\nu(\lambda x)-J_\nu(\lambda x)\frac{d}{dx}J_\nu(\mu x)\right\}dx$$

$$=(\mu^2-\lambda^2)\int_0^l xJ_\nu(\lambda x)J_\nu(\mu x)\,dx$$

or

$$\int_0^l\frac{d}{dx}\left\{x\left[J_\nu(\mu x)\frac{d}{dx}J_\nu(\lambda x)-J_\nu(\lambda x)\frac{d}{dx}J_\nu(\mu x)\right]\right\}dx$$

$$=(\mu^2-\lambda^2)\int_0^l xJ_\nu(\lambda x)J_\nu(\mu x)\,dx$$

Evaluation of the integrals yields

$$\left\{x\left[J_\nu(\mu x)\frac{d}{dx}J_\nu(\lambda x)-J_\nu(\lambda x)\frac{d}{dx}J_\nu(\mu x)\right]\right\}_0^l$$

$$=(\mu^2-\lambda^2)\int_0^l xJ_\nu(\lambda x)J_\nu(\mu x)\,dx \tag{7}$$

We shall show that for $\nu>-1$ and $\lambda=\alpha/l$, $\mu=\beta/l$, the left-hand side of Equation (7) will vanish. We note that, using Equation (3), we can write

$$J_\nu(\lambda x)=\frac{\left(\dfrac{\lambda x}{2}\right)^\nu}{\Gamma(\nu+1)}+x^{\nu+2}P_\lambda(x) \tag{8}$$

$$\frac{d}{dx}J_\nu(\lambda x)=\frac{\nu}{x}\frac{\left(\dfrac{\lambda x}{2}\right)^\nu}{\Gamma(\nu+1)}+x^{\nu+1}Q_\lambda(x)$$

where $P_\lambda(x)$ and $Q_\lambda(x)$ are power series. Using these formulae we find that

$$xJ_\nu(\mu x)\frac{\mathrm{d}}{\mathrm{d}x}J_\nu(\lambda x)-xJ_\nu(\lambda x)\frac{\mathrm{d}}{\mathrm{d}x}J_\nu(\mu x)$$

$$=\left[\frac{(\mu x)^\nu}{2^\nu\Gamma(\nu+1)}+x^{\nu+2}P_\mu(x)\right]\left[\nu\frac{(\lambda x)^\nu}{2^\nu\Gamma(\nu+1)}+x^{\nu+2}Q_\lambda(x)\right]$$

$$-\left[\frac{(\lambda x)^\nu}{2^\nu\Gamma(\nu+1)}+x^{\nu+2}P_\lambda(x)\right]\left[\nu\frac{(\mu x)^\nu}{2^\nu\Gamma(\nu+1)}+x^{\nu+2}Q_\mu(x)\right]$$

$$=x^{2\nu+2}R_1(x)+x^{2\nu+4}R_2(x)$$

where $R_1(x)$ and $R_2(x)$ are power series. When $\nu>-1$ the last expression will vanish when $x=0$. Substituting $\lambda=\alpha/l$, $\mu=\beta/l$ in Equation (7), we obtain

$$\int_0^l xJ_\nu\left(\frac{\alpha}{l}x\right)J_\nu\left(\frac{\beta}{l}x\right)\mathrm{d}x=\frac{l^2}{\beta^2-\alpha^2}\left[\alpha J_\nu(\beta)\frac{\mathrm{d}}{\mathrm{d}\alpha}J_\nu(\alpha)-\beta J_\nu(\alpha)\frac{\mathrm{d}}{\mathrm{d}\beta}J_\nu(\beta)\right] \tag{9}$$

The orthogonality property follows directly from (9) for the above values of $\alpha$ and $\beta$. In the third case, the products $\alpha\,\mathrm{d}J_\nu(\alpha)/\mathrm{d}\alpha$ and $\beta\,\mathrm{d}J_\nu(\beta)/\mathrm{d}\beta$ of the right-hand side of (9) must be replaced by $-hJ_\nu(\alpha)$ and $-hJ_\nu(\beta)$.

Consider now Equation (9) and let us pass to the limit as $\beta\to\alpha$:

$$\left\|J_\nu\left(\frac{\alpha}{l}x\right)\right\|^2=\int_0^l xJ_\nu^2\left(\frac{\alpha}{l}x\right)\mathrm{d}x$$

$$=\lim_{\beta\to\alpha}\frac{l^2}{\beta^2-\alpha^2}\left[\alpha J_\nu(\beta)J_\nu'(\alpha)-\beta J_\nu(\alpha)J_\nu'(\beta)\right]$$

By l'Hôpital's rule we have

$$\left\|J_\nu\left(\frac{\alpha}{l}x\right)\right\|^2=\frac{l^2}{2\alpha}\left[\alpha J_\nu'(\alpha)J_\nu'(\alpha)-J_\nu(\alpha)J_\nu'(\alpha)-\alpha J_\nu(\alpha)J_\nu''(\alpha)\right] \tag{10}$$

Next, using the identity

$$z^2J_\nu''(z)+zJ_\nu'(z)+(z^2-\nu^2)J_\nu(z)\equiv 0$$

we obtain

$$-J_\nu''(\alpha)=\frac{1}{\alpha}J_\nu'(\alpha)+\left(1-\frac{\nu^2}{\alpha^2}\right)J_\nu(\alpha)$$

Substituting this expression for $J_\nu''(\alpha)$ into (10), we obtain

$$\left\|J_\nu\left(\frac{\alpha}{l}x\right)\right\|^2 = \frac{l^2}{2}\left\{[J_\nu'(\alpha)]^2 + \left(1-\frac{\nu^2}{\alpha^2}\right)J_\nu^2(\alpha)\right\} \qquad (11)$$

We note that the square of the norm in the range $[a, b]$ of any function $Y_\nu(\lambda x)$ satisfying Equation (6) can be calculated from the formula

$$\int_a^b xY_\nu^2(\lambda x)\,\mathrm{d}x = \frac{z^2}{2\lambda^2}\left\{\left[\frac{\mathrm{d}Y_\nu(z)}{\mathrm{d}z}\right]^2 + \left(1-\frac{\nu^2}{z^2}\right)Y_\nu^2(z)\right\}\Bigg|_{a\lambda}^{b\lambda}$$

where $z = \lambda x$. To prove this we use the obvious relationship

$$\int_a^b xY_\nu(\lambda x)Y_\nu(\mu x)\,\mathrm{d}x = \left\{\frac{x}{\mu^2-\lambda^2}\left[Y_\nu(\mu x)\frac{\mathrm{d}}{\mathrm{d}x}Y_\nu(\lambda x) - Y_\nu(\lambda x)\frac{\mathrm{d}}{\mathrm{d}x}Y_\nu(\mu x)\right]\right\}_a^b$$

and then pass to the limit as $\mu \to \lambda$. In evaluating the limit of the right-hand side, we must use l'Hôpital's rule in Equation (6) just as in the calculation of $\left\|J_\nu\left(\frac{\alpha}{l}x\right)\right\|$.

**11.1.4** We saw in Section 11.1.3 that the orthogonality of Bessel functions is connected with the zeros of the Bessel functions and their derivatives. The zeros of the Bessel functions have the following properties.

1. All the zeros are simple, except perhaps for $z = 0$.
2. All the zeros of $J_\nu(z)$ with real subscript $\nu > -1$ are real.
3. Any Bessel function has an infinite number of zeros.

The validity of these statements follows from the following theorems.

*Theorem 1* The zeros of any cylindrical function are simple, except, possibly for $z = 0$.

*Theorem 2* All the zeros of the Bessel functions $J_\nu(z)$ with real $\nu > -1$ are real.

*Theorem 3* Any real cylindrical function has an infinite number of zeros.

*Proof of Theorem 1* Suppose that $z_0 \neq 0$ is a zero of order $n \geqslant 2$ of the solution $y_\nu(z)$ of Equation (1), which is not identically zero. We then have $y_\nu(z_0) = y_\nu'(z_0) = 0$. In view of the uniqueness of the solution of Cauchy's problem for Equation (1), $y_\nu(z) \equiv 0$. This is in conflict with our hypothesis and, therefore, the zero must be simple.

*Corollary* All the zeros of cylindrical functions are isolated zeros.

*Proof* The fact that $z_0 = 0$ is an isolated zero follows from the definition of Bessel function $J_\nu(z)$ and from the theorem in Section 11.1.2. In fact, any cylindrical function $y_\nu(z)$ can be written in the form

$$y_\nu(z) = C_1 J_\nu(z) + C_2 N_\nu(z)$$

where $N_\nu(z)$ is a solution of (1), which is linearly independent of $J_\nu(z)$. If $y_\nu(z) = C_1 J_\nu(z)$, and using Equation (3), we can write this function in the form $y_\nu(z) = C_1 z^\nu \varphi(z)$, where $\varphi(0) \neq 0$. Hence, it follows that $z = 0$ is an isolated zero of $y_\nu(z)$. If $y_\nu(z) = C_1 J_\nu(z) + C_2 N_\nu(z)$, where $C_2 \neq 0$, then $z = 0$ cannot be a zero of this function since $N_\nu(z)$ becomes infinite at $z = 0$, in accordance with the theorem in Section 2.

Let $z_0 \neq 0$ be a limit point of a sequence of zeros $\{z_n\} \to z_0$ of the cylindrical function $y_\nu(z)$ and $y_\nu(z_0) = 0$. It is clear that

$$y_\nu'(z_0) = \lim_{n\to\infty} \frac{y_\nu(z_n) - y_\nu(z_0)}{z_n - z_0} = 0$$

We therefore have $y_\nu(z_0) = 0$ and $y_\nu'(z_0) = 0$. This means that $z_0$ is a zero of multiplicity two, i.e. we have arrived at a contradiction, by Theorem 1.

It is evident that the above statement is equivalent to the following; in any bounded domain of the variable $z$, any cylindrical function $y_\nu(z)$ has a finite number of zeros.

To prove the second theorem, we must first establish the following lemma.

*Lemma* If $\alpha = re^{i\varphi}$ is a root of the equation $J_\nu(\gamma) = 0$, then the conjugate number $\bar{\alpha} = re^{-i\varphi}$ is a root of the same equation.

*Proof* The Bessel function can be written in the following form

$$J_\nu(z) = z^\nu \sum_{k=0}^{\infty} b_k z^{2k} \quad \text{where} \quad b_k = \frac{(-1)^k}{2^{\nu+2k}\Gamma(k+\nu+1)\Gamma(k+1)}$$

We then have

$$J_\nu(re^{i\varphi}) = r^\nu e^{i\nu\varphi} \sum_{k=0}^{\infty} b_k r^{2k} e^{i2k\varphi}$$

$$= r^\nu e^{i\nu\varphi} \left( \sum_{k=0}^{\infty} b_k r^{2k} \cos 2k\varphi + i \sum_{k=0}^{\infty} b_k r^{2k} \sin 2k\varphi \right)$$

or

$$\left.\begin{aligned} J_\nu(re^{i\varphi}) &= r^\nu e^{i\nu\varphi} [A_\nu(r, \varphi) - iD_\nu(r, \varphi)] \\ J_\nu(re^{-i\varphi}) &= r^\nu e^{-i\nu\varphi} [A_\nu(r, \varphi) + iD_\nu(r, \varphi)] \end{aligned}\right\} \tag{12}$$

where

$$A_\nu(r, \varphi) = \sum_{k=0}^{\infty} b_k r^{2k} \cos 2k\varphi, \quad D_\nu(r, \varphi) = \sum_{k=0}^{\infty} b_k r^{2k} \sin 2k\varphi$$

The lemma follows immediately from (12). In fact, let $\alpha = re^{i\varphi}$ be a root of the equation $J_\nu(\gamma) = 0$. Hence,

$$J_\nu(\alpha) = J_\nu(re^{i\varphi}) = r^\nu e^{i\nu\varphi} [A_\nu(r, \varphi) + iD_\nu(r, \varphi)] = 0$$

Consequently, $A_\nu(r, \varphi) = D_\nu(r, \varphi) = 0$. It also follows that $J_\nu(\overline{\alpha}) = 0$.

*Proof of Theorem 2* Let $\alpha = re^{i\varphi}$ be a root of the Bessel function $J_\nu(z)$. In accordance with the above lemma, $\overline{\alpha} = re^{-i\varphi}$ is also a zero of this function. In view of the orthogonality property, we then have

$$\int_0^l x J_\nu\left(\frac{\alpha}{l} x\right) J_\nu\left(\frac{\overline{\alpha}}{l} x\right) dx = 0$$

or

$$0 = \int_0^l x J_\nu\left(\frac{r}{l} x e^{i\varphi}\right) J_\nu\left(\frac{r}{l} x e^{-i\varphi}\right) dx$$

$$= \int_0^l x \left(\frac{rx}{l}\right)^{2\nu} [A_\nu^2(r, \varphi) + D_\nu^2(r, \varphi)] \, dx \tag{13}$$

In deriving this expression, we have used Equation (12). However, the integrand in the last integral is continuous and is not identically zero. Consequently, the integral itself cannot be equal

to zero. It follows that the assumed existence of complex roots of the Bessel function leads to a contradiction.

Theorem 3 will be proved in Section 11.3.9.

*Remark* It follows from (3) that the roots of the equation $J_\nu(z) = 0$ are arranged symmetrically with respect to $z = 0$ on the $z$ plane. Using the identities

$$\frac{d}{dz}\left[\frac{J_\nu(z)}{z^\nu}\right] \equiv \frac{-J_{\nu+1}(z)}{z^\nu} \text{ and } \frac{d}{dz}[z^\nu J_\nu(z)] \equiv z^\nu J_{\nu-1}(z) \quad (14)$$

which can be verified directly, the reader can readily prove the following theorem.

*Theorem 4* The functions $J_\nu(z)$ and $J_{\nu+1}(z)$ do not have common roots (except possibly for $z = 0$).

If we carry out the differentiation in (14) and eliminate $J'_\nu(z)$, we obtain the recurrence relation

$$J_{\nu+1}(z) \equiv -J_{\nu-1}(z) + \frac{2\nu}{z} J_\nu(z) \quad (15)$$

If $\nu$ is equal to an integer $n$, then we can use the recurrence relation to express all the functions $J_n(z)$, $n \geqslant 2$ in terms of $J_0(z)$ and $J_1(z)$.

Using the above theorems, the positive roots of the equation $J_\nu(\gamma) = 0$, where $\nu$ is a real number, can be arranged in an increasing sequence

$$\gamma_1 < \gamma_2 < \dots < \gamma_m < \gamma_{m+1} < \dots$$

It is evident that these roots are functions of the subscript $\nu$, i.e.

$$\gamma_m = \gamma_m(\nu)$$

*Theorem 5* $\gamma_m(\nu)$ are increasing functions of the variable $\nu$, if $\nu > 0$.

*Proof* For any fixed $m$ we have

$$J_\nu[\gamma_m(\nu)] \equiv 0 \quad (16)$$

Differentiating this identity with respect to $\nu$ and omitting, for simplicity, the subscript $m$, we obtain

$$J'_\nu(z)|_{z=\gamma} \frac{d\gamma}{d\nu} + \frac{\partial}{\partial\nu} J_\nu(z)|_{z=\gamma} = 0 \quad (17)$$

The prime will represent the derivative with respect to $z$. Next, from the first formula in (14) we have

$$zJ'_\nu(z)-\nu J_\nu(z) \equiv -zJ_{\nu+1}(z)$$

and hence, using (16), we obtain

$$J'_\nu(z)|_{z=\gamma} = -J_{\nu+1}(\gamma) \tag{18}$$

Returning now to (17), we have

$$\frac{d\gamma}{d\nu} = \frac{1}{J_{\nu+1}(\gamma)}\frac{\partial}{\partial\nu}J_\nu(z)|_{z=\gamma} \tag{19}$$

Differentiating the identity

$$\frac{d}{dz}[zJ'_\nu(z)]+\left(z-\frac{\nu^2}{z}\right)J_\nu(z) \equiv 0 \tag{20}$$

with respect to $\nu$, we obtain

$$\frac{d}{dz}\left[z\frac{\partial}{\partial\nu}J'_\nu(z)\right]-\frac{2\nu}{z}J_\nu(z)+\left(z-\frac{\nu^2}{z}\right)\frac{\partial}{\partial\nu}J_\nu(z) \equiv 0 \tag{21}$$

If we now multiply (20) and (21) by $(\partial/\partial\nu)J_\nu(z)$ and $J_\nu(z)$, respectively, and subtract one from the other, we have, after integrating with respect to $z$ between 0 and $\gamma$,

$$\int_0^\gamma \frac{d}{dz}\left\{z\left[J'_\nu(z)\frac{\partial}{\partial\nu}J_\nu(z)-J_\nu(z)\frac{\partial}{\partial\nu}J'_\nu(z)\right]\right\}dz = -2\nu\int_0^\gamma \frac{J_\nu^2(z)}{z}dz$$

or

$$\left\{zJ'_\nu(z)\frac{\partial}{\partial\nu}J_\nu(z)-zJ_\nu(z)\frac{\partial}{\partial\nu}J'_\nu(z)\right\}_0^\gamma = -2\nu\int_0^\gamma \frac{1}{z}J_\nu^2(z)dz$$

Since

$$J'_\nu(z) = b_2z^{\nu-1}+z^{\nu+1}P_2(z), \quad J_\nu(z) = b_1z^\nu+z^{\nu+2}P_1(z)$$

$$\frac{\partial}{\partial\nu}J_\nu(z) = b_3z^\nu\ln z+C_3z^\nu+z^{\nu+2}\ln z\,P_3(z)+z^{\nu+2}P_4(z)$$

$$\frac{\partial}{\partial\nu}J'_\nu(z) = b_4z^{\nu-1}\ln z+C_4z^{\nu-1}+z^{\nu+1}\ln z\,P_5(z)+z^{\nu+1}P_6(z)$$

where $P_k(z)$ are power series, $k = 1, 2, 3, 4, 5, 6$, the substitution of the lower limit of integration $z = 0$ yields zero. Therefore, using (16) and (18) we obtain

$$2\nu\int_0^\gamma \frac{1}{z}J_\nu^2(z)dz = \gamma J_{\nu+1}(\gamma)\frac{\partial}{\partial\nu}J_\nu(z)|_{z=\gamma}$$

Using this formula and Equation (19), we have

$$\frac{d\gamma_m}{d\nu} = \frac{2\nu}{\gamma_m J_{\nu+1}^2(\gamma_m)} \int_0^{\gamma_m} \frac{1}{z} J_\nu^2(z)\, dz > 0$$

This proves the theorem.

It is readily verified that

$$J_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \sin z \quad \text{and} \quad J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}} \cos z$$

**11.1.5** *Example* Consider the cooling of a uniform infinite rod of circular cross-section (radius $R$), whose surface is maintained at zero temperature. The initial temperature at internal points in the rod is given to be $\varphi(r)$.

The mathematical formulation of the problem is as follows: it is required to find the solution $u(r, t)$ of the equation $\nabla^2 u = (1/a^2) u_t$ for $t > 0$ and $0 \leqslant r < R$ which satisfies the following initial and boundary conditions:

$$u(r, 0) = \varphi(r), \quad u(R, t) = 0, \quad |u(0, t)| < \infty$$

*Solution* Separating the variables so that $u(r, t) = \Phi(r)\, \Psi(t)$, we find that

$$\Phi'' + \frac{1}{r}\Phi' + \lambda\Phi = 0, \quad \Phi(R) = 0, \quad |\Phi(0)| < \infty$$

$$\Psi(t) = C e^{-\lambda a^2 t} \text{ for } \lambda > 0$$

The general solution of the equation for $\Phi(r)$ can be written in the form

$$\Phi(r) = A J_0(\sqrt{\lambda}\, r) + B N_0(\sqrt{\lambda}\, r)$$

where $N_0 \sqrt{\lambda}\, r$ is a solution of the equation for $\Phi(r)$ which is linearly independent of $J_0(\sqrt{\lambda} r)$.

By the theorem in Section 11.1.2 $N_0(\sqrt{\lambda}\, r)$ is unbounded in the neighbourhood of $r = 0$. Since the required solution must be bounded, we have $B = 0$.

Consequently, $\Phi(r) = A J_0(\sqrt{\lambda}\, r)$. It is clear that we can set $A = 1$. From the boundary condition for $r = R$, we find the equation for the eigenvalues

$$J_0(\mu) = 0, \quad \mu = \sqrt{\lambda}\, R$$

By Theorems 1–3 (Section 11.1.4) this equation has an infinite number of simple real roots

$$\mu_1 < \mu_2 < \dots < \mu_n < \dots$$

These can be used to define the eigenvalues

$$\lambda_n = \frac{\mu_n^2}{R^2}.$$

and the eigenfunctions for the problem $J_0\left(\frac{\mu_n}{R} r\right)$.

We shall suppose that this set of eigenfunctions is complete and that the functions $\varphi(r)$ can be expanded into a series in terms of the eigenfunctions $J_0\left(\frac{\mu_n}{R} r\right)$.

The solution of the original problem will be sought in the form of the series

$$u(r, t) = \sum_{n=1}^{\infty} C_n \mathrm{e}^{-a^2 \frac{\mu_n^2}{R^2} t} J_0\left(\frac{\mu_n}{R} r\right)$$

The coefficients $C_n$ will be found using the initial conditions and the orthogonality of Bessel functions

$$u(r, 0) = \varphi(r) \equiv \sum_{n=1}^{\infty} C_n J_0\left(\frac{\mu_n}{R} r\right)$$

Let us multiply this identity by $rJ_0\left(\frac{\mu_k}{R} r\right)$ and integrate the result with respect to $r$ within the range $[0, R]$. Using the orthogonality of the Bessel functions and the formulae for the square of their norm, we obtain

$$\int_0^R r\,\varphi(r) J_0\left(\frac{\mu_k}{R} r\right) \mathrm{d}r = C_k \frac{R^2}{2} [J_0'(\mu_k)]^2$$

and, consequently,

$$C_k = \frac{2}{R^2 [J_0'(\mu_k)]^2} \int_0^R r\,\varphi(r) J_0\left(\frac{\mu_k}{R} r\right) \mathrm{d}r$$

*Remark* For an approximate solution of the problem, it is sufficient to retain only a few of the first terms in the series, for example

$$u(r, t) \approx C_1 \, e^{-a^2 \frac{\mu_1^2}{R^2} t} J_0\left(\frac{\mu_1}{R} r\right) + C_2 \, e^{-\frac{a^2 \mu_2^2}{R^2} t} J_0\left(\frac{\mu^2}{R} r\right)$$

$\mu_1$ and $\mu_2$ can be found from tabulated values of $J_0(x)$. They are

$$\mu_1 = 2.4048, \quad \mu_2 = 5.5201$$

## 11.2 HANKEL FUNCTIONS

**11.2.1** Cylindrical functions of the second class are constructed as follows. We seek the solution of Equation (1) in the form of the contour integral

$$w(z) = \int_C K(z, \xi) \, v(\xi) \, d\xi \tag{22}$$

where $K(z, \xi)$ is a given function and $v(\xi)$ is an unknown function. Substituting for $w(z)$ into the left-hand side of (1), we obtain

$$L[w] = \int_C \{z^2 K_{zz} + z K_z + z^2 K - \nu^2 K\} v(\xi) \, d\xi \tag{23}$$

We suppose that the contour $C$ and the function $K(z, \xi)$ are chosen so that all the above operations can be carried out.

The function $K(z, \xi)$ is chosen to be the solution of the equation

$$z^2 K_{zz} + z K_z + z^2 K + K_{\xi\xi} = 0 \tag{24}$$

so that $L(w)$ can be written in the form

$$L[w] = -\int_C (\nu^2 K + K_{\xi\xi}) v(\xi) \, d\xi$$

$$= -\int_C K \{v'' + \nu^2 v\} \, d\xi + \{K v' + K_\xi v\}_A^B$$

This formula is obtained by double integration by parts of the second term; $A$ and $B$ are the end-points of the contours of integration.

Let us now take the function $K(z, \xi)$ to be $(1/\pi) e^{-iz \sin \xi}$ and let us take $v(\xi)$ as a solution of the equation

$$v'' + \nu^2 v = 0$$

for example $e^{i\nu\xi}$. The contour $C$ is chosen so that all the above operations can be performed and so that the expression $Kv' - K_\xi v$

is equal to zero at the ends of the contour $C$, i.e. at the points $A$ and $B$. We then have

$$w(z) = \frac{1}{\pi} \int_C e^{-iz \sin \xi + i\nu\xi} \, d\xi \tag{25}$$

**11.2.2** If $C_1$ and $C_2$ are the end-points of the contour $C$ (Fig. 11.1), we obtain the following two cylindrical functions

$$\begin{aligned} H_\nu^{(1)}(z) &= \frac{1}{\pi} \int_{C_1} e^{-iz \sin \xi + i\nu\xi} \, d\xi \\ H_\nu^{(2)}(z) &= \frac{1}{\pi} \int_{C_2} e^{-iz \sin \xi + i\nu\xi} \, d\xi \end{aligned} \tag{26}$$

which are called Hankel functions.

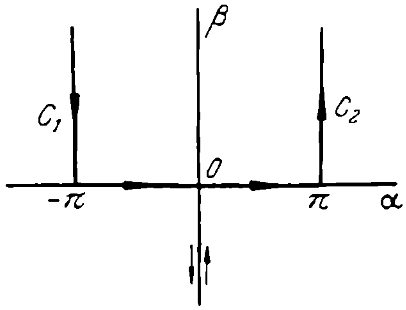



*Fig. 11.1*

So far the Hankel functions have been introduced in a purely formal manner. It is therefore necessary to show that the functions $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$, defined by (26), are, in fact, the solutions of (1), i.e. they have first- and second-order derivatives and that when they are substituted into Equation (1) the differentiation (both first- and second-order) can be carried out under the integral sign. It is also necessary to show that for the above choice of $C_1$ and $C_2$, the expression $Kv' - K_z v$ vanishes at the ends of these contours.

We shall now establish some of the properties of Hankel functions.

1. Hankel functions are defined and continuous in the domain $\mathrm{Re}\, z > 0$.

To show this, it is sufficient to establish the uniform convergence of the integrals which define the Hankel functions for $\mathrm{Re}\, z \geqslant \delta > 0$, where $\delta$ is any positive number. To be specific, consider the function $H_\nu^{(1)}(z)$. On the upper part of the contour $C_1$

$$\xi = -\pi + i\beta \; (\beta > 0), \quad \sin\xi = -i \sinh \beta$$

On the lower part of $C_1$

$$\xi = i\beta(\beta < 0), \qquad \sin\xi = i \sinh\beta$$

Consequently, on these parts of $C_1$ the functions $e^{-\delta\sinh\beta - s\beta + \pi q}$ and $e^{\delta\sinh\beta - \beta s}$ will be the majorants of the modulus of the integrand ($\nu = s + iq$), respectively. At the same time, the integrals of these functions $\int_0^\infty e^{-\delta\sinh\beta - s\beta + \pi q}\,d\beta$ and $\int_{-\infty}^0 e^{\delta\sinh\beta - s\beta}\,d\beta$ will converge. Hence the original integral over $C_1$ will converge uniformly for $\operatorname{Re} z \geqslant \delta > 0$. The uniform convergence of the integrals

$$\int_{C_p} K_z v\,d\xi, \quad \int_{C_p} K_{zz} v\,d\xi, \quad \int_{C_p} K_{\xi\xi} v\,d\xi \quad (p = 1, 2)$$

may be established in a similar way.

2. Hankel functions are analytic for $\operatorname{Re} z > 0$. To establish this, we note that $\int_L H_\nu^{(1)}(z)\,dz$, evaluated over any piecewise-smooth contour $L$ in the region $\operatorname{Re} z \geqslant \delta > 0$, is zero, since

$$\int_L H_\nu^{(1)}(z)\,dz = \int_{C_1}\int_L e^{-iz\sin\xi + i\nu\xi}\,dz\,d\xi = 0$$

The change of the order of integration is justified because the integral over $C_1$ converges uniformly and the function $e^{-iz\sin\xi}$ is analytic everywhere. By Morera's theorem it follows that $H_\nu^{(1)}(z)$ is an analytic function. The fact that $H_\nu^{(2)}(z)$ is analytic can be shown in a similar way.

Since the integrals $\int_{C_p} K_z v\,d\xi$ and $\int_{C_p} K_{zz} v\,d\xi$ converge uniformly for $\operatorname{Re} z \geqslant \delta$ ($\delta > 0$), it follows that when one evaluates the derivatives of Hankel functions one can differentiate under the integral sign.

3. The following limits are valid:

$$\left.\begin{aligned} \lim_{\xi = -\pi + i\beta \to -\pi + i\infty} \{K(z, \xi)v'(\xi) - K_\xi(z, \xi)v(\xi)\} &= 0 \\ \lim_{\xi = i\beta \to -i\infty} \{K(z, \xi)v'(\xi) - K_\xi(z, \xi)v(\xi)\} &= 0 \end{aligned}\right\} \operatorname{Re} z > 0 \qquad (27)$$

Let us prove the first of these. On the upper part of $C_1$

$$|K(z, \xi)v'(\xi)| = |e^{-iz\sin\xi + i\nu\xi} i\nu| = |\nu| e^{-x\sinh\beta - s\beta + \pi q} \to 0, \quad \beta \to \infty$$

$$\begin{aligned} |K_\xi(z, \xi)v(\xi)| &= |-iz\cos\xi\, e^{-iz\sin\xi - i\nu\xi}| \\ &= |z|\cosh\beta\, e^{-x\sinh\beta - s\beta + \pi q} \to 0, \quad \beta \to \infty. \end{aligned}$$

This proves the first relationship.

The second relationship can be established in a similar way. It follows that the Hankel functions are the solutions of Equation (1) and are analytic in the half-plane $\operatorname{Re} z > 0$.

4. The Hankel functions are linearly independent. To prove this, it is sufficient to show the existence of a sequence $\{z_n\}$ in the range in which the Hankel functions are defined, in which one of the functions tends to infinity and the other is bounded. Consider Equations (26). On the upper and lower parts of $C_1$ we have $\xi = -\pi+i\beta$ and, correspondingly, $\xi = -i\beta$ $(\beta > 0)$. The integrals over these parts of $C_1$ can therefore be written in the form

$$\frac{-i}{\pi}\int_0^\infty e^{-z\sinh\beta-\nu\beta}e^{-i\nu\pi}\,d\beta \quad\text{and}\quad \frac{-i}{\pi}\int_0^\infty e^{-z\sinh\beta+\nu\beta}\,d\beta$$

Consequently, the Hankel function $H_\nu^{(1)}(z)$ can be written in the form

$$H_\nu^{(1)}(z) = \frac{-i}{\pi}e^{-i\pi\nu}\int_0^\infty e^{-z\sinh\beta-\nu\beta}\,d\beta - \frac{i}{\pi}\int_0^\infty e^{-z\sinh\beta+\nu\beta}\,d\beta + \frac{1}{\pi}\int_{-\pi}^0 e^{-iz\sin\xi+i\nu\xi}\,d\xi$$

or

$$H_\nu^{(1)}(z) = \frac{-i}{\pi}e^{-i\nu\pi}\int_0^\infty e^{-z\sinh\beta-\nu\beta}\,d\beta - \frac{i}{\pi}\int_0^\infty e^{-z\sinh\beta+\nu\beta}\,d\beta + \frac{1}{\pi}\int_0^\pi e^{iz\sin\xi-i\nu\xi}\,d\xi \tag{26$_1$}$$

Similarly,

$$H_\nu^{(2)}(z) = \frac{i}{\pi}e^{i\pi\nu}\int_0^\infty e^{-z\sinh\beta-\nu\beta}\,d\beta + \frac{i}{\pi}\int_0^\infty e^{-z\sinh\beta+\nu\beta}\,d\beta + \frac{1}{\pi}\int_0^\pi e^{-iz\sin\xi+i\nu\xi}\,d\xi \tag{26$_2$}$$

As $z$ tends to infinity, over the sequence $z_n = x_0+iy_n$, where $x_0$ is fixed, $x_0 > 0$, $y_n > 0$ and $y_n \to \infty$, the first two integrals in $(26_1)$ and $(26_2)$ remain bounded since

$$\left|\int_0^\infty e^{-z_n\sinh\beta\mp\nu\beta}\,d\beta\right| \leqslant \int_0^\infty e^{-x_0\sinh\beta\mp\nu\beta}\,d\beta$$

The integral

$$\int_0^\pi e^{-iz_n \sin\xi - i\nu\xi}\, d\xi = \int_0^\pi e^{-y_n \sin\xi} e^{ix_0 \sin\xi - i\nu\xi}\, d\xi$$

will also be bounded as $y_n \to \infty$. Consequently, $H_\nu^{(1)}(z_n)$ remains bounded for $z_n \to \infty$. The integral

$$\int_0^\pi e^{-iz_n \sin\xi + i\nu\xi}\, d\xi = \int_0^\pi e^{y_n \sin\xi} e^{-ix_0 \sin\xi + i\nu\xi}\, d\xi$$

in $(26_2)$ tends to infinity as $y_n \to +\infty$, since $\sin\xi > 0$ within the range of integration. Consequently, $H_\nu^{(2)}(z_n) \to \infty$ as $z_n \to \infty$. This proves the linear independence of Hankel functions.

**11.2.3** Direct evaluation will show that the following recurrence relations are valid:

$$H_{\nu+1}^{(k)}(z) + H_{\nu-1}^{(k)}(z) \equiv \frac{2\nu}{z} H_\nu^{(k)}(z) \qquad (k = 1, 2) \tag{28}$$

$$H_{\nu+1}^{(k)}(z) - H_{\nu-1}^{(k)}(z) \equiv -2 \frac{d}{dz} H_\nu^{(k)}(z) \quad (k = 1, 2) \tag{29}$$

In fact,

$$H_{\nu+1}^{(k)}(z) + H_{\nu-1}^{(k)}(z) = \frac{1}{\pi} \int_{C_k} e^{-iz\sin\xi} (e^{i(\nu+1)\xi} + e^{i(\nu-1)\xi})\, d\xi$$

$$= \frac{1}{\pi} \int_{C_k} e^{-iz\sin\xi + i\nu\xi} (e^{i\xi} + e^{-i\xi})\, d\xi$$

$$= \frac{2}{\pi} \int_{C_k} e^{-iz\sin\xi + i\nu\xi} \cos\xi\, d\xi$$

Integrating by parts, we obtain

$$-\frac{2}{i\pi z} \int_{C_1} e^{i\nu\xi}\, d(e^{-iz\sin\xi}) = -\frac{2}{i\pi z} e^{-iz\sin\xi + i\nu\xi} \Big|_{\xi=-\pi+i\infty}^{\xi=-i\infty}$$

$$+ \frac{2\nu}{z}\frac{1}{\pi} \int_{C_1} e^{-iz\sin\xi + i\nu\xi}\, d\xi = \frac{2\nu}{z}\frac{1}{\pi} \int_{C_1} e^{-iz\sin\xi + i\nu\xi}\, d\xi = \frac{2\nu}{z} H_\nu^{(1)}(z)$$

since the substitution of the limits into the integrand yields zero. The procedure for $H_\nu^{(2)}(z)$ is the same. Next,

$$H_{\nu+1}^{(k)}(z)-H_{\nu-1}^{(k)}(z)=\frac{1}{\pi}\int_{C_k}e^{-iz\sin\xi+i\nu\xi}2i\sin\xi\,d\xi$$

On the other hand,

$$2\frac{d}{dz}H_\nu^{(k)}(z)=\frac{2}{\pi}\int_{C_k}e^{-iz\sin\xi+i\nu\xi}(-i\sin\xi)\,d\xi$$

Consequently, Equation (29) is also valid.

For Re $z>0$ we have

$$J_\nu(z)=\frac{H_\nu^{(1)}(z)+H_\nu^{(2)}(z)}{2} \tag{30}$$

Since the functions $J_\nu(z)$ and $H_\nu^{(k)}(z)$ $(k=1,2)$ are analytic for Re $z>0$, it is sufficient to show that (30) is valid for $z=x>0$.

*Proof* Let $j_\nu(z)$ represent the right-hand side of (30). We then have

$$j_\nu(x)=\frac{1}{2\pi}\int_{C_0}e^{-ix\sin\xi+i\nu\xi}d\xi \tag{31}$$

where $C_0$ is the contour shown in Fig. 11.2.

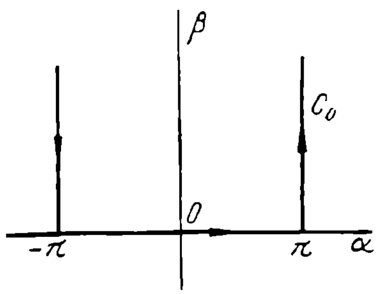



*Fig. 11.2*

Let us substitute

$$a=\frac{x}{2}e^{-i(\xi-\pi)}$$

into this integral. The half-line $(-\pi+i\infty,-\pi)$ of $C_0$ transforms into the half-line $(+\infty,x/2)$ on the real axis, and the half-line $(\pi,\pi+i\infty)$ into the same half-line $(x/2,+\infty)$ on the real axis, but oriented in the opposite direction; the segment $[-\pi,\pi]$ transforms into the circle $|a|=x/2$.

Under this transformation, therefore, $C_0$ will be transformed into the contour $\gamma$ (see Chapter 10), which is traversed in the opposite direction. Consequently,

$$j_\nu(x) = \frac{-i}{2\pi}\int\limits_\gamma e^{\frac{x^2}{4a}-a} e^{i\pi\nu}\left(\frac{x}{2}\right)^\nu \frac{da}{a^{\nu+1}}$$
$$= \frac{-e^{i\pi\nu}}{2\pi} i\left(\frac{x}{2}\right)^\nu \int\limits_\gamma e^{-a}a^{-\nu-1} e^{\frac{x^2}{4a}} da$$

Expanding $e^{x^2/4a}$ in a Laurent expansion in powers of $a$ and integrating term by term, we obtain

$$j_\nu(x) = \frac{ie^{i\nu\pi}}{2\pi}\left(\frac{x}{2}\right)^\nu \sum_{k=0}^{\infty}\left(\frac{x}{2}\right)^{2k}\frac{1}{k!}\int\limits_\gamma e^{-a}a^{-k-\nu-1}\,da$$
$$= \sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{x}{2}\right)^{2k+\nu}}{\Gamma(k+\nu+1)\,\Gamma(k+1)} = J_\nu(x)$$

where we have used Equation (7) of Chapter 10.
Therefore,

$$J_\nu(x) = \frac{1}{2}\,[H_\nu^{(1)}(x)+H_\nu^{(2)}(x)]$$

We have obtained the integral representation of the Bessel functions

$$J_\nu(x) = \frac{1}{2\pi}\int\limits_{C_0} e^{-iz\sin\xi+i\nu\xi}\,d\xi \tag{32}$$

If we split this into three integrals, we obtain

$$J_\nu(z) = \frac{i}{2\pi}\int\limits_0^\infty e^{iz\sin(i\beta)+\pi i\nu-\nu\beta}\,d\beta - \frac{i}{2\pi}\int\limits_0^\infty e^{iz\sin(i\beta)-i\pi\nu-\nu\beta}\,d\beta$$
$$+\frac{1}{2\pi}\int\limits_{-\pi}^{\pi} e^{-iz\sin\alpha+i\nu\alpha}\,d\alpha \qquad (\xi = \alpha+i\beta)$$

or

$$J_\nu(z) = \frac{i}{2\pi}(e^{i\pi\nu}-e^{-i\nu\pi})\int\limits_0^\infty e^{-z\sinh\beta-\nu\beta}\,d\beta + \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} e^{-iz\sin\alpha+i\nu\alpha}\,d\alpha$$

or

$$J_\nu(z) = \frac{-\sin \pi\nu}{\pi} \int_0^\infty e^{-z \sinh \beta - \nu\beta} d\beta + \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-iz \sin \alpha + i\nu\alpha} d\alpha \quad (33)$$

In particular, for $\nu = n$ (an integer) we have

$$J_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-iz \sin \alpha + in\alpha} d\alpha \quad (34)$$

It follows immediately that

$$|J_n(z)| \leqslant \cosh y \quad (z = x+iy)$$

and

$$|J_n(x)| \leqslant 1$$

**11.2.4** The functions

$$N_\nu(z) = \frac{H_\nu^{(1)}(z) - H_\nu^{(2)}(z)}{2i} \quad (35)$$

are called Neumann's functions. They are analytic for Re $z > 0$ since the functions $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ are analytic in this region. It is clear that $N_\nu(z)$ is a solution of (1).

It is evident from Equations (30) and (35) that

$$\begin{aligned} H_\nu^{(1)}(z) &= J_\nu(z) + iN_\nu(z) \\ H_\nu^{(2)}(z) &= J_\nu(z) - iN_\nu(z) \end{aligned} \quad (35_1)$$

It is readily shown that the functions $J_\nu(z)$ and $N_\nu(z)$ are linearly independent. If this were not so, there would exist a constant $C$ such that for Re $z > 0$ we would have $N_\nu(z) \equiv CJ_\nu(z)$. However, from $(35_1)$ we would then have

$$H_\nu^{(1)}(z) = (1+iC)J_\nu(z) \quad \text{and} \quad H_\nu^{(2)}(z) = (1-iC)J_\nu(z)$$

and hence $H_\nu^{(2)}(z) \equiv DH_\nu^{(1)}(z)$, where $D = (1-iC)/(1+iC)$, which contradicts the linear independence of the Hankel functions proved above.

Therefore, the functions $J_\nu(z)$ and $N_\nu(z)$ regarded as functions of $H_\nu^{(1)}$ and $H_\nu^{(2)}$ form a fundamental system of solutions of Equation

(1). From ($35_1$) it follows that the recurrence relations (28) and (29) of this chapter are valid also for $J_\nu(z)$ and $N_\nu(z)$:

$$J_{\nu+1}(z)+J_{\nu-1}(z) \equiv \frac{2\nu}{z} J_\nu(z), \qquad J_{\nu+1}(z)-J_{\nu-1}(z) \equiv -2J'_\nu(z)$$

$$N_{\nu+1}(z)+N_{\nu-1}(z) \equiv \frac{2\nu}{z} N_\nu(z), \qquad N_{\nu+1}(z)-N_{\nu-1}(z) \equiv -2N'_\nu(z)$$

## 11.3 ASYMPTOTIC REPRESENTATIONS OF CYLINDRICAL FUNCTIONS

**11.3.1** In many problems encountered in physics it is necessary to evaluate the steady-state solutions. Mathematically, this leads to the consideration of functions for large values of the arguments, i.e. to the asymptotic behaviour of these functions. When studying the asymptotic behaviour of functions in a region $\mathcal{E}$, they are replaced by simpler functions with similar properties. Most frequently, the simpler functions are taken in the form

$$c_0+\frac{c_1}{z}+\frac{c_2}{z^2}+ \dots +\frac{c_n}{z^n}$$

*Definition* The series $c_0+c_1/z+ \dots +c_n/z^n+ \dots$ is called the asymptotic expansion of a function $f(z)$ in a region $\mathcal{E}$ containing sequences converging at infinity if

$$\lim_{\substack{z\to\infty \\ z\in\mathcal{E}}} z^n \left\{ f(z)-\sum_{k=0}^{n} \frac{c_k}{z^k} \right\} = 0 \text{ where } n = 0, 1, 2, \dots$$

The following notation is used:

$$f(z) \sim c_0+\frac{c_1}{z}+ \dots +\frac{c_n}{z^n}+ \dots$$

or

$$f(z) = c_0+\frac{c_1}{z}+ \dots +\frac{c_n}{z^n}+o\left(\frac{1}{z^{n+1}}\right)$$

It is readily shown that if the asymptotic expansion exists, then it is unique.

In point of fact, it follows from the definition that when $n = 0$, $\lim_{z\to\infty} \{f(z)-c_0\} = 0$, and hence $c_0 = \lim_{z\to\infty} f(z)$. For $n = 1$ we have

$$\lim_{z\to\infty} z\left\{f(z)-c_0-\frac{c_1}{z}\right\} = 0$$

and hence

$$c_1 = \lim_{z\to\infty} z\,\{f(z)-c_0\}$$

$$\cdot\ \cdot \qquad\qquad \cdot\ \cdot\ \cdot$$

$$c_n = \lim_{z\to\infty} z^n\left\{f(z)-\sum_{k=0}^{n-1}\frac{c_k}{z^k}\right\}$$

$$\cdot\ \cdot$$

However, different functions may have the same asymptotic expansions. In fact, if

$$f(x) \sim c_0+\frac{c_1}{x}+\ldots+\frac{c_n}{x^n}+\ldots$$

then also

$$f(x)+e^{-x} \sim c_0+\frac{c_1}{x}+\ldots+\frac{c_n}{x^n}+\ldots \qquad (\text{for } x > 0)$$

If

$$f(z) = \frac{\psi(z)}{\varphi(z)} = c_0+\frac{c_1}{z}+\ldots+\frac{c_n}{z^n}+o\left(\frac{1}{z^{n+1}}\right)$$

then

$$\psi(z) = \varphi(z)\left[c_0+\frac{c_1}{z}+\ldots+\frac{c_n}{z^n}+o\left(\frac{1}{z^{n+1}}\right)\right]$$

will be called the asymptotic representation of $\psi(z)$.

**11.3.2** We shall now find the asymptotic representation of the error integral

$$\Phi(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$$

for large $x > 0$. It is clear that

$$\Phi(x) = 1 - \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2}\, dt$$

It is therefore sufficient to find the asymptotic representation of the functions $f(x) = \int_x^{\infty} e^{-t^2}\, dt$.

We have

$$e^{x^2} f(x) = \int_x^{\infty} e^{x^2 - t^2}\, dt = \frac{1}{2} \int_x^{\infty} \frac{e^{x^2-t^2}}{t}\, d(t^2) = \frac{-1}{2} \int_x^{\infty} \frac{d(e^{x^2-t^2})}{t}$$

Integrating by parts, we obtain

$$e^{x^2} f(x) = \frac{1}{2x} - \frac{1}{2^2 x^3} + \frac{1 \cdot 3}{2^3 x^5} - \dots$$
$$+ (-1)^{n-1} \frac{1 \cdot 3 \cdot 5 \dots (2n-3)}{2^n x^{2n-1}} + R_n(x)$$

For the remainder

$$R_n(x) = (-1)^n \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2^n} \int_x^{\infty} \frac{e^{x^2-t^2}}{t^{2n}}\, dt$$

we obtain

$$|R_n(x)| < \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2^{n+1} x^{2n+1}} = o\left(\frac{1}{x^{2n+1}}\right)$$

Consequently,

$$f(x) = e^{-x^2} \left\{ \frac{1}{2x} - \frac{1}{2^2 x^3} + \frac{1 \cdot 3}{2^3 x^5} - \dots \right.$$
$$\left. + \frac{(-1)^{n-1}(2n-3)!!}{2^n x^{2n-1}} + o\left(\frac{1}{x^{2n+1}}\right) \right\}$$

and

$$\Phi(x) = 1 - \frac{2}{\sqrt{\pi}} e^{-x^2} \left\{ \frac{1}{2x} - \frac{1}{2^2 x^3} + \frac{1 \cdot 3}{2^3 x^5} - \dots \right.$$
$$\left. + \frac{(-1)^{n-1}(2n-3)!!}{2^n x^{2n-1}} + o\left(\frac{1}{x^{2n+1}}\right) \right\}$$

**11.3.3** One of the most widely used methods for obtaining asymptotic representations is the method of steepest descents. This is based on the fact that for large values of the variable $x$ the magnitude of the integral

$$f(x) = \int_C \psi(\xi) e^{x\varphi(\xi)}\,d\xi$$

is determined mainly by the part $C_{II}$ of the contour of integration $C$, on which $|e^{x\varphi(\xi)}| = e^{x\,\mathrm{Re}\,\varphi(\xi)}$ is large in comparison with the values of this modulus on the remainder of $C$. The integral over $C_{II}$ is estimated more readily the shorter this region and the steeper the fall of the quantity $x$ Re $\varphi(\xi)$. In the method of steepest descents, one tries to deform the contour of integration $C$ into the most convenient form $C$ in the above sense. By Cauchy's theorem this deformation does not affect the value of the integral provided we remain in the region in which $\varphi(\xi)$ and $\psi(\xi)$ are analytic. Since the function $\varphi(\xi) = u(\alpha, \beta) + i\,v(\alpha, \beta)$, $\xi = \alpha + i\beta$ is analytic, the direction of the most rapid variation of $u(\alpha, \beta)$ is the same as the direction of the line $v(\alpha, \beta) = $ const. The deformed contour $\tilde{C}_{II}$ must contain the point $\xi_0 = \alpha_0 + i\beta_0$, at which $u(\alpha, \beta)$ reaches its maximum value (among the values of this function on $\tilde{C}$).

It is readily shown that $\varphi'(\xi_0) = 0$. In fact, the derivative of $u(\alpha, \beta)$ along the line $\tilde{C}$, taken at the point $\xi_0$, is equal to zero since, at this point, the function $u(\alpha, \beta)$ reaches its maximum value (along $\tilde{C}$).

Next, $\left.\dfrac{\partial v}{\partial s}\right|_{\xi=\xi_0} = 0$, since in the neighbourhood of the point $\xi = \xi_0$, we have $v = $ const along $\tilde{C}$. Therefore,

$$\varphi'(\xi_0) = \left.\frac{\partial u}{\partial s}\right|_{\xi=\xi_0} + i\left.\frac{\partial v}{\partial s}\right|_{\xi=\xi_0} = 0$$

The point $\xi_0$ for the surface $u = u(\alpha, \beta)$ is, of course, a saddle point.

Therefore, in applying the method of steepest descents to the asymptotic form of the integral $\int_C \psi(\xi) e^{x\varphi(\xi)} d\xi$, the path of integration $C$ must be deformed into $\tilde{C}$ which passes through $\xi_0$ where $\varphi'(\xi_0) = 0$, and in the neighbourhood of this point coincides with the line $v(\alpha, \beta) = $ const $= v(\alpha_0, \beta_0)$.

**11.3.4** We shall use the method to deduce the asymptotic behaviour of the Hankel functions $H_\nu^{(1)}(x)$ and $H_\nu^{(2)}(x)$ for large values of $x$ ($x > 0$).

Let us establish the following theorem.

*Theorem* Any real solution of the equation

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\left(1-\frac{\nu^2}{x^2}\right)y = 0 \quad (\operatorname{Im}\nu = 0)$$

has an asymptotic representation of the form

$$y(x) = \frac{A_0}{\sqrt{x}}\sin(x+\delta_0)\left[1+O\left(\frac{1}{x}\right)\right]$$

where $A_0$ and $\delta_0$ are constants.

*Proof* Consider the function $y_1(x)$ as defined by

$$y(x) = \frac{y_1(x)}{\sqrt{x}}$$

The differential equation for $y_1(x)$ is

$$y_1''+\left(1-\frac{\nu^2-\frac{1}{4}}{x^2}\right)y_1 = 0 \tag{36}$$

For large $x$ this equation is not very different from

$$\frac{d^2w}{dx^2}+w = 0$$

whose general solution is

$$w = A\sin(x+\delta) \tag{37}$$

where $A$ and $\delta$ are constants.

Therefore, for large values of $x$ we shall seek the solution of (36) in the form

$$y_1(x) = A(x)\sin[x+\delta(x)]$$

where $A(x)$ and $\delta(x)$ are the required functions.

Since there are two required functions, and they are related by only one condition, the requirement that $A(x)\sin[x+\delta(x)]$ must satisfy (36), we can subject them to one further condition. We shall choose this in such a way that the derivative of $y_1(x)$ is evaluated as if $A(x)$ and $\delta(x)$ were constants. Since

$$y_1' = A\cos(x+\delta)+A\delta'\cos(x+\delta)+A'\sin(x+\delta)$$

we shall suppose that

$$A\delta'\cos(x+\delta)+A'\sin(x+\delta) \equiv 0 \tag{38}$$

We then have

$$y_1' = A\cos(x+\delta) \tag{39}$$

Evaluating the derivative $y_1''$ and substituting this into (36), we obtain

$$A'\cos(x+\delta)-A\left(\delta'+\frac{\gamma}{x^2}\right)\sin(x+\delta) \equiv 0 \tag{40}$$

where

$$\gamma = \nu^2-\frac{1}{4}$$

Eliminating $A$ and $A'$ from (38) and (40), we obtain

$$\delta'(x) = \frac{-\gamma}{x^2}\sin^2(x+\delta) \tag{41}$$

and hence

$$\delta(b) = \delta(x)-\int_x^b \frac{\gamma}{\xi^2}\sin^2[\xi+\delta(\xi)]\,d\xi \tag{42}$$

For a fixed $x$ and $b \to \infty$, the right-hand side of (42) tends to a limit. Consequently, the left-hand side will also have a limit:

$$\lim_{b\to\infty} \delta(b) = \delta_0$$

We therefore have

$$\delta(x) = \delta_0+\int_x^\infty \frac{\gamma}{\xi^2}\sin^2[\xi+\delta(\xi)]\,d\xi$$

However,

$$\left|\int_x^\infty \frac{\gamma}{\xi^2}\sin^2(\xi+\delta)\,d\xi\right| \leqslant |\gamma|\int_x^\infty \frac{d\xi}{\xi^2} = \frac{|\gamma|}{x} = O\left(\frac{1}{x}\right)$$

and therefore

$$\delta(x) = \delta_0+O\left(\frac{1}{x}\right)$$

From (38) and (41) we have

$$(\ln A)' = \frac{A'}{A} = \frac{\gamma}{2x^2}\sin 2(x+\delta)$$

and consequently

$$\ln A(b) = \ln A(x) + \frac{\gamma}{2}\int_x^b \frac{\sin 2(\xi+\delta)}{\xi^2}\,d\xi$$

Repeating the argument given for $\delta(x)$ and $\delta(b)$, we arrive at the conclusion that the limit

$$\lim_{b\to\infty} \ln A(b) = \ln A_0$$

exists and

$$\ln A(x) = \ln A_0 + O\left(\frac{1}{x}\right)$$

Consequently,

$$A(x) = A_0\left[1+O\left(\frac{1}{x}\right)\right]$$

Therefore,

$$\begin{aligned} y_1(x) &= A_0\left[1+O\left(\frac{1}{x}\right)\right]\sin\left[x+\delta_0+O\left(\frac{1}{x}\right)\right] \\ &= A_0 \sin(x+\delta_0)\left[1+O\left(\frac{1}{x}\right)\right] \end{aligned}$$

and

$$y(x) = \frac{A_0}{\sqrt{x}}\sin(x+\delta_0)\left[1+O\left(\frac{1}{x}\right)\right]$$

**11.3.5** Let us return now to the consideration of Hankel functions. To be specific, consider

$$H_\nu^{(1)}(x) = \frac{1}{\pi}\int_{C_1} e^{-ix\sin\xi+i\nu\xi}\,d\xi = \frac{1}{\pi}\int_{C_1} e^{i\nu\xi}e^{-ix\sin\xi}\,d\xi$$

where $\varphi(\xi) = -i\sin\xi$. The saddle points $\xi_0$ can be found from the equation

$$\varphi'(\xi_0) = -i\cos\xi_0 = 0, \qquad \xi_0 = \frac{\pi}{2}(2k+1)$$

where $k$ is an integer. Since the contour $C_1$ lies in the strip $-\pi \leqslant \operatorname{Re} \xi \leqslant 0$, we shall be interested only in $\xi_0 = -\pi/2$:

$$\varphi(\xi) = -i \sin \xi = -i \sin (\alpha + i\beta) = -i \sin \alpha \cosh \beta + \cos \alpha \sinh \beta$$

Therefore, $v(\alpha, \beta) = -\sin \alpha \cosh \beta$. In the neighbourhood of the point $\xi_0$ the deformed contour $\tilde{C}_1$ has the equation

$$-\sin \alpha \cosh \beta = -\sin \alpha_0 \cosh \beta_0 = 1$$

or

$$\sin \alpha \cosh \beta = -1$$

The direction of this curve at the point $\xi_0 = -\pi/2$ is determined by the angular coefficient

$$\left.\frac{d\beta}{d\alpha}\right|_{\alpha=-\frac{\pi}{2}} = \left.\frac{1}{\sin \alpha}\right|_{\alpha=\frac{-\pi}{4}} = -1$$

Therefore the contour $\tilde{C}_1$ can be taken as the broken line $(+\infty, \xi_1, \xi_2, \xi_3, \xi_4, -\infty)$, shown in Fig. 11.3.

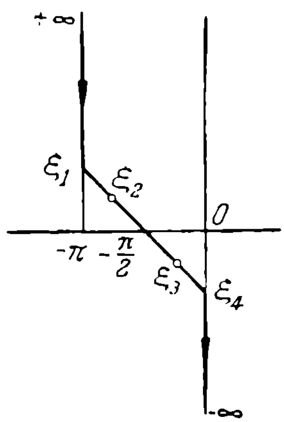



*Fig. 11.3*

In accordance with the above discussion we can split this contour into three parts: $\tilde{C}_{\mathrm{I}}(+\infty, \xi_1, \xi_2)$, $\tilde{C}_{\mathrm{II}}(\xi_2, \xi_3)$ and $\tilde{C}_{\mathrm{III}}(\xi_3, \xi_4, -\infty)$. We shall assume that

$$\xi_2 = \alpha_2 + i\beta_2 = -\frac{\pi}{2} + x^{-\sigma} e^{i\frac{3}{4}\pi}$$

$$\xi_3 = \alpha_3 + i\beta_3 = -\frac{\pi}{2} + x^{-\sigma} e^{-i\frac{\pi}{4}}$$

where $\sigma$ is a fixed number $1/4 < \sigma < 1/2$.

**11.3.6** Let us now estimate the integrals over $\tilde{C}_{\text{I}}$ and $\tilde{C}_{\text{III}}$. It is clear that

$$\left|\int_{\tilde{C}_{\text{I}}} e^{-ix\sin\xi+i\nu\xi}\,d\xi\right| \leqslant \left|\int_{\tilde{C}_{\text{I}}} e^{-ix(\sin\alpha\cosh\beta+i\cos\alpha\sinh\beta)+i\nu(\alpha+i\beta)}\,d\xi\right|$$

$$\leqslant \int_{C_{\text{I}}} e^{x\cos\alpha\sinh\beta-\nu\beta}|d\xi|$$

$$= e^{x\cos\alpha_2\sinh\beta_2}\int_{\tilde{C}_{\text{I}}} e^{x(\cos\alpha\sinh\beta-\cos\alpha_2\sinh\beta_2)-\nu\beta}|d\xi|$$

$$= e^{x\cos\alpha_2\sinh\beta_2}\Bigg\{\int_{(+\infty,\,\xi_1)} e^{x(\cos\alpha\sinh\beta-\cos\alpha_2\sinh\beta_2)-\nu\beta}\,d\beta$$

$$+\int_{(\xi_1,\,\xi_2)} e^{x(\cos\alpha\sinh\beta-\cos\alpha_2\sinh\beta_2)-\nu\beta}|d\xi|\Bigg\}$$

Each of these two integrals is bounded since $\cos\alpha\sinh\beta-\cos\alpha_2\sinh\beta_2<0$ for $\xi\neq\xi_2$ and decreases with increasing $x$. Consequently,

$$\int_{\tilde{C}_1} = O(e^{x\cos\alpha_2\sinh\beta_2}) = O(e^{-\frac{1}{2}x^{1-2\sigma}})$$

since

$$\alpha_2 = -\frac{\pi}{2}-\frac{1}{x^\sigma\sqrt{2}}, \quad \beta_2 = \frac{1}{x^\sigma\sqrt{2}}$$

and

$$\cos\alpha_2\sinh\beta_2 = \left(\frac{-1}{x^\sigma\sqrt{2}}+\frac{x^{-3\sigma}}{2\sqrt{2}\cdot 3!}-\dots\right)\left(\frac{x^{-\sigma}}{\sqrt{2}}+\frac{x^{-3\sigma}}{3!\,2\sqrt{2}}+\dots\right)$$

$$= \frac{-1}{2x^{2\sigma}}+\frac{1}{6!\,x^{6\sigma}}+\dots$$

Similarly,

$$\left|\int_{\tilde{C}_{\text{III}}} e^{-ix\sin\xi+i\nu\xi}\,d\xi\right| \leqslant e^{x\cos\alpha_3\sinh\beta_3}$$

$$\times\Bigg\{\int_{(\xi_3,\,\xi_4)} e^{x(\cos\alpha\sinh\beta-\cos\alpha_3\sinh\beta_3)-\nu\beta}|d\xi|$$

$$+\int_{(\xi_4,\,-\infty)} e^{x(\cos\alpha\sinh\beta-\cos\alpha_3\sinh\beta_3)-\nu\beta}\,d\beta\Bigg\} = O(e^{-\frac{x^{1-2\sigma}}{2}})$$

**11.3.7** Let us estimate the integral over $\tilde{C}_{II}$. We have $\xi = -\pi/2 + re^{i3\pi/4}$ in the range $(\xi_2, -\pi/2)$, $0 \leqslant r \leqslant x^{-\sigma}$, and $\xi = -\pi/2 + re^{-i\pi/4}$ in the range $(-\pi/2, \xi_3)$, $0 \leqslant r \leqslant x^{-\sigma}$. These two formulae can be combined into one:

$$\xi = -\frac{\pi}{2} + re^{-i\frac{\pi}{4}}$$

where $-x^{-\sigma} \leqslant r \leqslant x^{-\sigma}$. Therefore,

$$\int\limits_{\tilde{C}_{II}} e^{-xi\sin\xi + i\nu\xi} d\xi = \int\limits_{-x^{-\sigma}}^{x^{-\sigma}} e^{ix\cos(re^{-i\frac{\pi}{4}}) - i\frac{\pi}{2}\nu + i\nu re^{-i\frac{\pi}{4}}} e^{-i\frac{\pi}{4}} dr$$

$$= e^{i\left(x - \nu\frac{\pi}{2} - \frac{\pi}{4}\right)} \int\limits_{-x^{-\sigma}}^{x^{-\sigma}} e^{ix[\cos(re^{-i\frac{\pi}{4}}) - 1] + i\nu re^{-i\frac{\pi}{4}}}$$

Expanding $\cos(re^{-i\pi/4})$ into a series in powers of $r$, we obtain

$$R = \int\limits_{-x^{-\sigma}}^{x^{-\sigma}} e^{ix[\cos(re^{-i\frac{\pi}{4}}) - 1] + i\nu re^{-i\frac{\pi}{4}}} dr$$

$$= \int\limits_{-x^{-\sigma}}^{x^{-\sigma}} e^{-\frac{xr^2}{2}} e^{m_1(x,r)} e^{im_2(x,r)} dr$$

where

$$m_1(x, r) = \frac{\nu r}{\sqrt{2}} + x\left[\frac{r^6}{6!} - \frac{r^{10}}{10!} + \frac{r^{14}}{14!} - \dots \right.$$
$$\left. + (-1)^{n+1}\frac{r^{2+4n}}{(2+4n)!} + \dots\right]$$

$$m_2(x, r) = \frac{\nu r}{\sqrt{2}} - x\left[\frac{r^4}{4!} - \frac{r^8}{8!} + \dots + (-1)^{n+1}\frac{r^{4n}}{(4n)!} + \dots\right]$$

or

$$R = \int\limits_{-x^{-\sigma}}^{x^{-\sigma}} e^{-\frac{xr^2}{2}} e^{m_1}(\cos m_2 + i\sin m_2) dr$$

Applying the mean value theorem to each of these two integrals, we obtain

$$R = \{e^{m_1(x, \theta_1 x^{-\sigma})}\cos m_2(x, \theta_1 x^{-\sigma})$$
$$+ ie^{m_1(x, \theta_2 x^{-\sigma})}\sin m_2(x, \theta_2 x^{-\sigma})\} \int\limits_{-x^{-\sigma}}^{x^{-\sigma}} e^{-\frac{xr^2}{2}} dr$$

As $x \to \infty$, the functions $m_k(x, \theta_j x^{-\sigma})$ $(k, j = 1, 2)$ tend to zero as $x^{-\sigma}$. Consequently,

$$R = \{1+[\rho_1(x)+i\rho_2(x)]x^{-\sigma}\} \int_{-x^{-\sigma}}^{x^{-\sigma}} e^{-\frac{xr^2}{2}} dr$$

where $\rho_1(x)$ and $\rho_2(x)$ are bounded as $x \to \infty$. Substituting $r\sqrt{x/2} = \gamma$, we obtain

$$R = \{1+(\rho_1+i\rho_2)x^{-\sigma}\} 2\sqrt{\frac{2}{x}} \int_0^{\sqrt{\frac{x}{2}}x^{-\sigma}} e^{-\gamma^2} d\gamma$$

Using the asymptotic representation of the error integral (see Section 11.3.2), we obtain

$$R = \sqrt{\frac{2}{x}} \{1+\rho(x)x^{-\sigma}\} \sqrt{\pi} \{1-O(e^{-\frac{1}{2}x^{1-2\sigma}})\}$$
$$= \sqrt{\frac{2\pi}{x}} \{1+\rho_3(x)x^{-\sigma}O(e^{-0.5x^{1-2\sigma}})\}$$

where

$$\rho(x) = \rho_1(x)+i\rho_2(x), \qquad \rho_3(x) = \rho(x)+x^{\sigma}O(e^{-0.5x^{1-2\sigma}})$$

and the modulus of $\rho_3(x)$ is bounded for large $x > 0$. Therefore, for large $x > 0$ we have

$$H_\nu^{(1)}(x) = \sqrt{\frac{2}{\pi x}} e^{i\left(x-\nu\frac{\pi}{2}-\frac{\pi}{4}\right)} \{1+\rho_3(x)x^{-\sigma}\} \tag{43}$$

Similarly, it can be shown that

$$H_\nu^{(2)}(x) = \sqrt{\frac{2}{\pi x}} e^{-i\left(x-\nu\frac{\pi}{2}-\frac{\pi}{4}\right)} \{1+\tilde{\rho}(x)x^{-\sigma}\} \tag{44}$$

where $\tilde{\rho}(x)$ is bounded as $x \to \infty$.

**11.3.8** From (43), (44), (30) and (35) we obtain

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x-\nu\frac{\pi}{2}-\frac{\pi}{4}\right) \{1+a_1(x)x^{-\sigma}\} \tag{45}$$

$$N_\nu(x) = \sqrt{\frac{2}{\pi x}} \sin\left(x-\nu\frac{\pi}{2}-\frac{\pi}{4}\right) \{1+a_2(x)x^{-\sigma}\} \tag{46}$$

$$a_1(x) = \frac{\rho_3(x)-\tilde{\rho}(x)}{2}, \qquad a_2(x) = \frac{1}{2i}\{\rho_3(x)-\tilde{\rho}(x)\}$$

Using the theorem on the asymptotic representation of real cylindrical functions (see Section 11.3.4) and the uniqueness of the asymptotic representation, we conclude that $a_1(x)x^{-\sigma}$ and $a_2(x)x^{-\sigma}$ are small quantities of the order of $1/x$. Consequently, $\rho(x)x^{-\sigma}$ and $\tilde{\rho}(x)x^{-\sigma}$ are small quantities of the same order.

Therefore, we obtain the following improved versions of (43)–(46):

$$H_\nu^{(1)}(x) = \sqrt{\frac{2}{\pi x}}\, e^{i\left(x-\nu\frac{\pi}{2}-\frac{\pi}{4}\right)}\left\{1+O\left(\frac{1}{x}\right)\right\} \tag{47}$$

$$H_\nu^{(2)}(x) = \sqrt{\frac{2}{\pi x}}\, e^{-i\left(x-\nu\frac{\pi}{2}-\frac{\pi}{4}\right)}\left\{1+O\left(\frac{1}{x}\right)\right\} \tag{48}$$

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}}\cos\left(x-\nu\frac{\pi}{2}-\frac{\pi}{4}\right)\left\{1+O\left(\frac{1}{x}\right)\right\} \tag{49}$$

$$N_\nu(x) = \sqrt{\frac{2}{\pi x}}\sin\left(x-\nu\frac{\pi}{2}-\frac{\pi}{4}\right)\left\{1+O\left(\frac{1}{x}\right)\right\} \tag{50}$$

These are, in fact, the required asymptotic representations.

We note that these formulae are valid for any $z$: $|z| \gg 1$, $|\arg z| < \pi-\delta$, where $\delta$ is an arbitrary small number.

**11.3.9** The validity of Theorem 3 of Section 11.1.4 follows from Equations (49) and (50) and from the linear independence of the functions $J_\nu(z)$ and $N_\nu(z)$. In fact, because of the linear independence of these functions, an arbitrary real cylindrical function $y_\nu(z)$ can be obtained from the formula

$$y_\nu(z) = D_1 J_\nu(z) + D_2 N_\nu(z)$$

Consequently, it will have the following asymptotic representation

$$y_\nu(z) = \sqrt{\frac{2}{\pi z}}\left[D_1\cos\left(z-\frac{\pi}{2}\nu-\frac{\pi}{4}\right)\right.$$
$$\left.+D_2\sin\left(z-\nu\frac{\pi}{2}-\frac{\pi}{4}\right)\right]\left\{1+O\left(\frac{1}{z}\right)\right\}$$

Substituting $D_1/D_2 = \tan\omega$, we obtain

$$y_\nu(z) = \sqrt{\frac{2}{\pi z}}\sqrt{D_1^2+D_2^2}\sin\left(z-\nu\frac{\pi}{2}-\frac{\pi}{4}+\omega\right)\left\{1+O\left(\frac{1}{z}\right)\right\}$$

Theorem 3 of Section 11.1.4 follows directly from this asymptotic representation, together with the result that the distance between

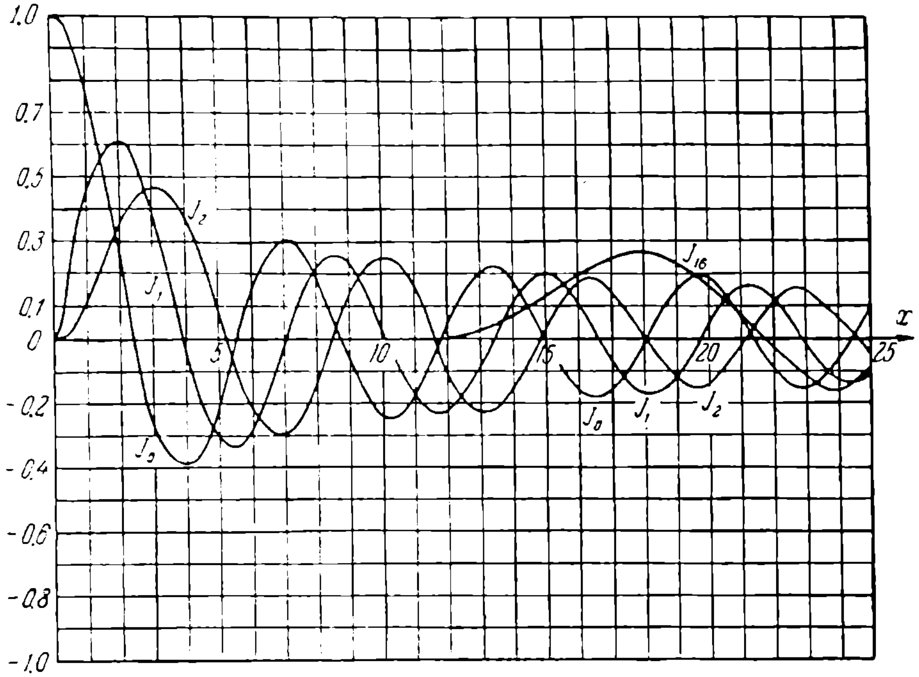



*Fig. 11.4*

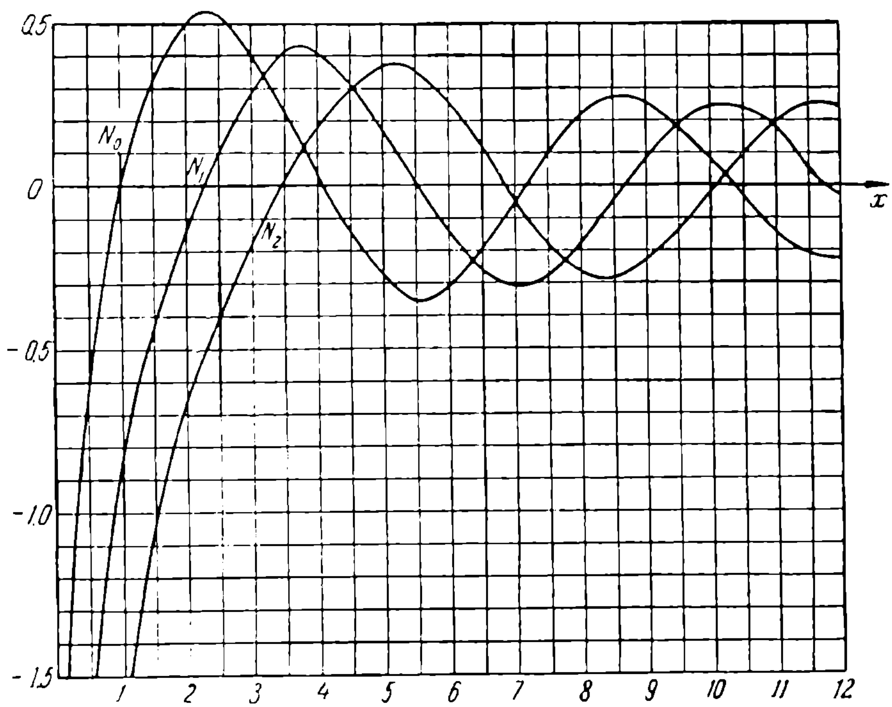



*Fig. 11.5*

two neighbouring zeros of the Bessel function $J_\nu(z)$ (and of the Neumann function) tends to $\pi$ as the absolute magnitudes of the zeros increase without bound. Plots of Bessel and Neumann functions are shown in Figs. 11.4 and 11.5. that

**11.3.10** Using the asymptotic behaviour of the $J_\nu(z)$, $J_{-\nu}(z)$, $N_\nu(z)$, $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$, one can readily show that

$$N_\nu(z) = \frac{J_\nu(z)\cos\nu\pi - J_{-\nu}(z)}{\sin\nu\pi}$$

and

$$H_\nu^{(1)}(z) = \frac{i}{\sin\nu\pi}[e^{-i\pi\nu}J_\nu(z) - J_{-\nu}(z)]$$

$$H_\nu^{(2)}(z) = \frac{-i}{\sin\nu\pi}[e^{i\pi\nu}J_\nu(z) - J_{-\nu}(z)]$$

If $\nu$ is an integer, the right-hand sides of these formulae are meaningless since they are of the form 0/0. The right-hand sides must then be regarded as the limits as $\nu \to n$. These formulae can be taken as the definitions of Neumann and Hankel functions.

## 11.4 THE FUNCTIONS $I_\nu(z)$, $K_\nu(z)$, etc.

**11.4.1** The third class of cylindrical functions can be defined as follows:

$$I_\nu(z) = i^{-\nu}J_\nu(iz) \tag{51}$$

$$K_\nu(z) = \frac{\pi}{2}H_\nu^{(1)}(iz)e^{i\frac{\pi}{2}(\nu+1)} \tag{52}$$

Since $J_\nu(z)$ and $H_\nu^{(1)}(z)$ are linearly independent, it follows that $I_\nu(z)$ and $K_\nu(z)$ are also linearly independent. The solutions of the equation

$$w'' + \frac{1}{z}w' - \left(1 + \frac{\nu^2}{z^2}\right)w = 0 \tag{53}$$

which is obtained from Equation (1) of this chapter by substituting $z = i\xi$. It is clear that

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{\nu+2k}}{\Gamma(k+\nu+1)\Gamma(k+1)} \tag{54}$$

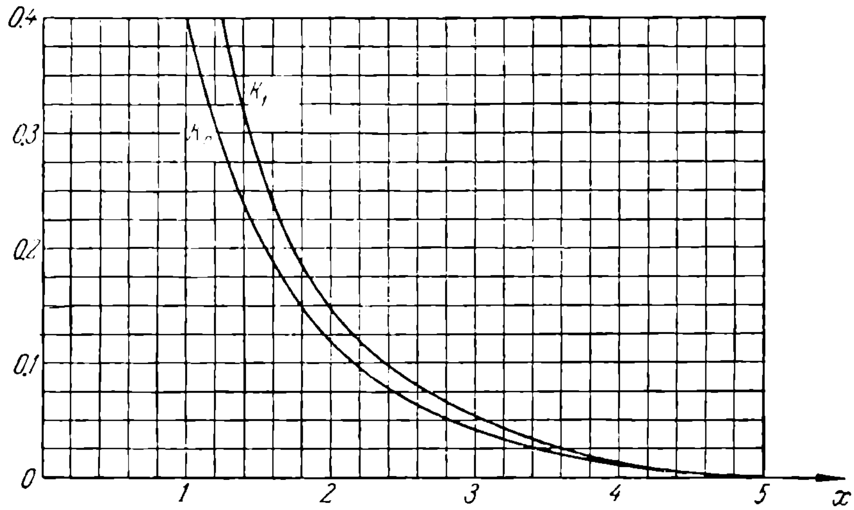



*Fig. 11.6*

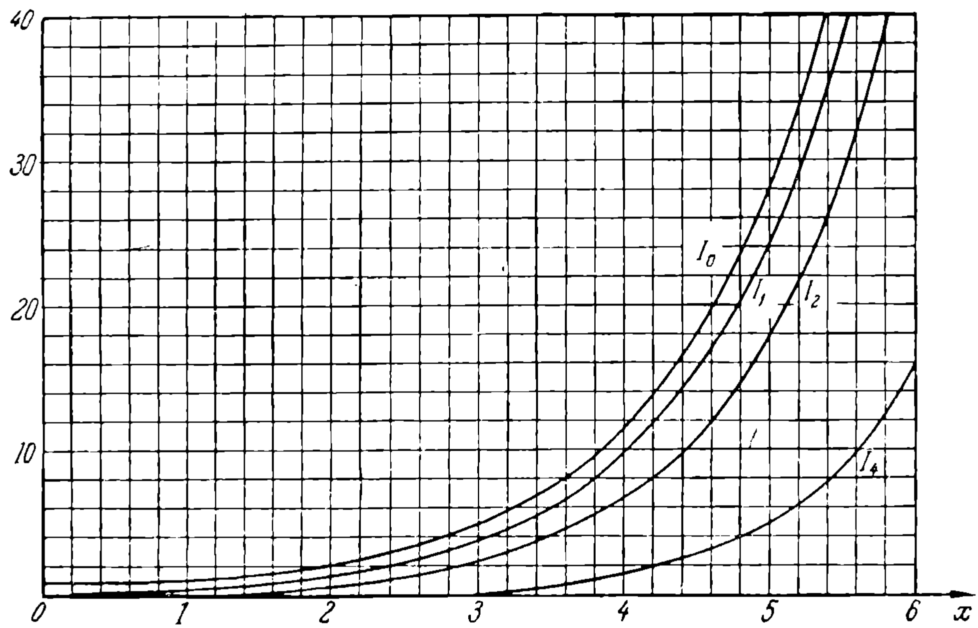



*Fig. 11.7*

Using Equations (47) and (49), it is readily shown that for large $|z|$ and $|\arg z| < \pi/2-\delta$,

$$I_\nu(z) = \sqrt{\frac{1}{2\pi z}}\, e^{z}\left\{1+O\left(\frac{1}{z}\right)\right\} \tag{55}$$

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}}\, e^{-z}\left\{1+O\left(\frac{1}{z}\right)\right\} \tag{56}$$

From the properties of the zeros of Bessel functions it follows that $I_\nu(z)$ has an infinite number of simple, purely imaginary zeros. At the point $z = 0$ the function $K_\nu(z)$ has a singularity of the form $z^{-\nu}$, $\nu \neq 0$, and at $\nu = 0$ it has a logarithmic singularity. There are recurrence relations for these functions which correspond to the formulae for $J_\nu(z)$ and $H_\nu^{(1)}(z)$. Plots of $K_\nu(x)$ and $I_\nu(x)$ are given in Figs. 11.6 and 11.7.

**11.4.2** The following cylindrical functions

$$\mathrm{ber}_n(z), \qquad \mathrm{bei}_n(z), \qquad \mathrm{ker}_n(z), \qquad \mathrm{kei}_n(z)$$

are also encountered in applications. For real values of the argument $x$, they are defined as follows:

$$\mathrm{ber}_n(x) = \mathrm{Re}[I_n(x\sqrt{-i})], \quad \mathrm{bei}_n(x) = \mathrm{Im}[I_n(x\sqrt{-i})] \tag{57}$$

$$\mathrm{ker}_n(x) = \mathrm{Re}[K_n(x\sqrt{-i})], \quad \mathrm{kei}_n(x) = \mathrm{Im}[K_n(x\sqrt{-i})] \tag{58}$$

and then analytically continued to the entire plane of the variable $z$.

These definitions can be used to derive properties analogous to the corresponding properties of the functions $I_\nu(z)$ and $K_\nu(z)$. The derivation of these properties is left to the reader.

It can be shown that

$$\mathrm{ber}_0(z) = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{z}{2}\right)^{4n}}{[(2n)!]^2} \tag{59}$$

$$\mathrm{bei}_0(z) = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{z}{2}\right)^{2n+1}}{[(2n+1)!]^2} \tag{60}$$

$$\operatorname{ber}_1(z) = \frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{(-1)^{E\left[\frac{n}{2}\right]} \left(\frac{z}{2}\right)^{2n+1}}{n!(n+1)!} \tag{61}$$

$$\operatorname{bei}_1(z) = \frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{(-1)^{E\left[\frac{n+1}{2}\right]} \left(\frac{z}{2}\right)^{2n+1}}{n!(n+1)!} \tag{62}$$

where $E[y]$ is the integral part of the number $y$.

**11.4.3** *Example 1* Determine the steady-state temperature distribution in a uniform circular cylindrical rod of length $h$ and radius $R$, whose ends are maintained at zero temperature and whose lateral surface is kept at a temperature $f(z)$.

The mathematical formulation of the problem is as follows. It is required to find the solutions $u(r, z)$ of the equation $\Delta u = 0$ for $0 \leqslant r < R$, $0 < z < h$ satisfying the boundary conditions

$$u(r, 0) = u(r, h) = 0, \qquad |u(0, z)| < \infty, \qquad u(R, z) = f(z)$$

We shall seek solutions satisfying the first three boundary conditions in the form $u = \Phi(r)\Psi(z)$. For the functions $\Phi(r)$ and $\Psi(z)$ we shall have the following problems:

$$\Psi'' + \lambda\Psi = 0, \quad \Psi(0) = \Psi(h) = 0 \quad (\text{consequently } \lambda > 0)$$

$$\Phi'' + \frac{1}{r}\Phi' - \lambda\Phi = 0, \qquad |\Phi(0)| < \infty \tag{63}$$

It is clear that

$$\Psi_n(z) = \sin\frac{n\pi}{h}z, \quad \lambda_n = \frac{\pi^2 n^2}{h^2}$$

The general solution of (63) (for $\lambda = \lambda_n$) can be written in the form

$$\Phi_n(r) = C_n I_0\left(\frac{\pi n}{h}r\right) + D_n K_0\left(\frac{\pi n}{h}r\right)$$

Since the function $K_0\left(\frac{\pi n}{h}r\right)$ becomes infinite at $r = 0$, we must set $D_n = 0$. Therefore,

$$\Phi_n(r) = C_n I_0\left(\frac{\pi n}{h}r\right)$$

The solution of the original problem will be sought in the form

$$u(r, z) = \sum_{n=1}^{\infty} C_n I_0\left(\frac{\pi n}{h} r\right) \sin\frac{\pi n}{h} z$$

where $C_n$ is determined by the last boundary condition (assuming that the function $f(z)$ can be expanded into a Fourier series in terms of the eigenfunctions $\sin (\pi n/h)z$ in the range $(0, h)$):

$$C_n = \frac{2}{h I_0\left(\frac{\pi n}{h} R\right)} \int_0^h f(z) \sin\frac{\pi n}{h} z\, dz$$

*Example 2* Determine the steady-state temperature distribution in an infinite uniform flat plate of thickness $h$ having a circular aperture of radius $R$ if the faces of the plate are maintained at zero temperature and the walls of the aperture are kept at a temperature $f(z)$.

The mathematical formulation of the problem is: it is required to find the solution $u(r, z)$ of the equation $\Delta u = 0$ in the region $R \leqslant r < \infty, 0 < z < h$ satisfying the boundary conditions

$$u(r, 0) = u(r, h) = 0, \qquad |u(\infty, z)| < \infty, \qquad u(R, z) = f(z)$$

We shall seek solutions satisfying the first three boundary conditions in the form $u = \Phi(r)\Psi(z)$. For the functions $\Phi(r)$ and $\Psi(z)$ we have the following problems:

$$\Psi'' + \lambda\Psi = 0, \quad \Psi(0) = \Psi(h) = 0 \quad (\text{consequently } \lambda > 0)$$

$$\Phi'' + \frac{1}{r}\Phi' - \lambda\Phi = 0, \qquad |\Phi(\infty)| < \infty \tag{64}$$

It is clear that

$$\Psi_n(z) = \sin\frac{\pi n}{h} z, \qquad \lambda_n = \frac{\pi^2 n^2}{h^2}.$$

The general solution of (64) (for $\lambda = \lambda_n$) can be written in the form

$$\Phi_n(r) = D_n I_0\left(\frac{\pi n}{h} r\right) + C_n K_0\left(\frac{\pi n}{h} r\right)$$

Since the required solution must be bounded in the region $R \leqslant r < \infty$ and the function $I_0\left(\frac{\pi n}{h} r\right)$ is not bounded in this range of $r$ [since for large values of $r$ it behaves as $\sqrt{h/2\pi^2 n r}\; e^{(\pi n/h)r}(1+O(1/r)]$, we must set $D_n = 0$. Hence

$$\Phi_n(r) = C_n K_0\left(\frac{\pi n}{h} r\right)$$

The solution of the original problem will therefore be sought in the form

$$u(r, z) = \sum_{n=1}^{\infty} C_n K_0\left(\frac{\pi n}{h} r\right) \sin \frac{\pi n}{h} z$$

where

$$C_n = \frac{2}{h K_0\left(\frac{\pi n}{h} R\right)} \int_0^h f(z) \sin \frac{\pi n}{h} z \, dz$$

## 11.5 AIRY FUNCTIONS

Many problems in physics, for example the problem involving the motion of a charged particle in a uniform electric field, lead to the equation

$$y'' - xy = 0 \tag{65}$$

Substituting

$$y(x) = \begin{cases} \sqrt{x}\, z(x) & \text{for } x \geqslant 0 \\ \sqrt{-x}\, z(x) & \text{for } x < 0 \end{cases}$$

we obtain the following differential equation:

$$z'' + \frac{1}{x} z'(x) - \left(\frac{1}{4x^2} + x\right) z = 0 \tag{66}$$

To derive the general solution of this equation, let us make the further substitution

$$t = \begin{cases} \frac{2}{3} x^{3/2} & \text{for } x \geqslant 0 \\ \frac{2}{3} (-x)^{3/2} = \frac{2}{3} |x|^{3/2} & \text{for } x < 0 \end{cases}$$

Equation (66) will then become

$$\frac{d^2z}{dt^2}+\frac{1}{t}\frac{dz}{dt}-\left[\frac{1/9}{t^2}+1\right]z=0 \quad \text{for } x \geqslant 0$$

$$\frac{d^2z}{dt^2}+\frac{1}{t}\frac{dz}{dt}+\left[1-\frac{1/9}{t^2}\right]z=0 \quad \text{for } x<0$$

These are the equations for cylindrical functions. Their general solutions can be written in the following form:

$$z(t)=C_1 I_{-1/3}(t)+C_2 I_{1/3}(t) \quad \text{for } x \geqslant 0$$

$$z(t)=D_1 J_{-1/3}(t)+D_2 J_{1/3}(t) \quad \text{for } x<0$$

Consequently, the general solution of (65) can be written in the form

$$y(x)=\begin{cases} \sqrt{x}\left[C_1 I_{-1/3}\left(\frac{2}{3}x^{3/2}\right)+C_2 I_{1/3}\left(\frac{2}{3}x^{3/2}\right)\right] & \text{for } x \geqslant 0 \\ \sqrt{|x|}\left[D_1 J_{-1/3}\left(\frac{2}{3}|x|^{3/2}\right)+D_2 J_{1/3}\left(\frac{2}{3}|x|^{3/2}\right)\right] & \text{for } x<0 \end{cases}$$

If we set

$$C_1=-C_2=D_1=D_2=\frac{1}{3}$$

and

$$C_1=C_2=D_1=-D_2=\frac{1}{3}$$

we obtain the Airy functions $Ai(x)$ and $Bi(x)$:

$$Ai(x)=\begin{cases} \frac{\sqrt{x}}{3}\left[I_{-1/3}\left(\frac{2}{3}x^{3/2}\right)-I_{1/3}\left(\frac{2}{3}x^{3/2}\right)\right] & \text{for } x \geqslant 0 \\ \frac{\sqrt{|x|}}{3}\left[J_{-1/3}\left(\frac{2}{3}|x|^{3/2}\right)+J_{1/3}\left(\frac{2}{3}|x|^{3/2}\right)\right] & \text{for } x<0 \end{cases}$$

$$Bi(x)=\begin{cases} \frac{\sqrt{x}}{3}\left[I_{-1/3}\left(\frac{2}{3}x^{3/2}\right)+I_{1/3}\left(\frac{2}{3}x^{3/2}\right)\right] & \text{for } x \geqslant 0 \\ \frac{\sqrt{|x|}}{3}\left[J_{-1/3}\left(\frac{2}{3}|x|^{3/2}\right)-J_{1/3}\left(\frac{2}{3}|x|^{3/2}\right)\right] & \text{for } x<0 \end{cases}$$

From the power-series definitions of $I_\nu(t)$ and $J_\nu(t)$ it follows that

$$Ai(x) \underset{x\to 0}{\to} \frac{1}{\sqrt[3]{18}\,\Gamma(2/3)}, \qquad Bi(x) \underset{x\to 0}{\to} \frac{1}{\sqrt[3]{18}\,\Gamma(2/3)}$$

Using the same procedure as in Section 11.3.5, we can readily show that

$$Ai(x) = \frac{1}{2\sqrt{\pi}} x^{-1/4} e^{-2/3 x^{3/2}} [1+O(x^{-3/2})] \quad \text{for } x \to +\infty$$

$$Ai(x) = \frac{1}{\sqrt{\pi}} x^{-1/4} \sin\left(\frac{2}{3}|x|^{3/2}+\frac{\pi}{4}\right)[1+O(|x|^{-7/4})] \quad \text{for } x \to -\infty$$

$$Bi(x) = \frac{\sqrt{3}}{\sqrt{\pi}} x^{-1/4} e^{2/3 x^{3/2}} [1+O(x^{-3/2})] \quad \text{for } x \to +\infty$$

$$Bi(x) = \frac{\sqrt{3}}{\sqrt{\pi}} x^{-1/4} \sin\left(\frac{2}{3}|x|^{3/2}+\frac{\pi}{4}\right)[1+O(|x|^{-7/4})] \quad \text{for } x \to -\infty$$

The Airy functions have been tabulated (see bibliography).

## PROBLEMS

1. Find the temperature of an infinite circular cylinder whose initial temperature is $u(r, 0) = A(1-r^2/R^2)$, and whose surface is maintained at zero temperature.

2. A homogeneous cylindrical conductor of radius $R$ is heated for a considerable time by a constant current $R$. Investigate the cooling of the conductor after the current has been switched off if the surface of the conductor loses heat in accordance with Newton's law and the surrounding medium is kept at zero temperature.

3. A constant magnetic field $H_0$ is switched on outside an infinite circular conducting cylinder $0 \leqslant r \leqslant R$, of conductivity $\sigma$, at time $t = 0$. The magnetic field is parallel to the axis of the cylinder. Find the magnetic field inside the cylinder if the current is initially zero. Determine the flux of magnetic induction through the cross-section of the cylinder.

4. Find the temperature distribution in a cylindrical tube $R_1 \leqslant r \leqslant R_2$ if a heat flux of density $q$ is introduced at time $t = 0$

through the outer surface of the tube whilst the internal surface is kept at zero temperature. The initial temperature is zero.

5. Determine the vibrations of a circular membrane with fixed edges under the action of a uniformly distributed load $Q = \text{const}$, applied on one side, beginning with $t = 0$.

6. Solve Problem 5 for (a) $Q = A \sin \omega t$, (b) $Q = A \cos \omega t$ and (c) the load $Q$ distributed over the ring $R_1 \leqslant r \leqslant R_2$ (consider also the case $R_1 = R_2$).

7. Determine the vibrations of a circular membrane $0 \leqslant r \leqslant R$, whose edge moves for $t > 0$ in accordance with the laws: (a) $u(R, t) = A \sin \omega t$, (b) $u(R, t) = A \cos \omega t$. There is no initial excitation.

8. Determine the vibrations of a circular membrane $0 \leqslant r \leqslant R$ with fixed edges under the action of a point impulse $P$, applied to the membrane at the point $(r_0, \varphi_0)$ at time $t = 0$.

9. Find the electrostatic potential inside a hollow cylinder of radius $R$ and height $h$ whose lower base ($z = 0$) and lateral surface are kept at the potential $V_0$ whilst the upper base is kept at the potential $V_1$.

10. A constant current $I$ enters through one end of a cylindrical conductor made of material having a conductivity $\sigma$ and leaves at the opposite end. Determine the potential distribution inside the conductor, assuming that the end contacts are discs of radius $R_1 < R$, where $R$ is the radius of the cylinder, and that the current is distributed in the conductor with constant density.

11. A thin wire, heated by constant current and liberating $Q$ units of heat per unit length is introduced into a cylindrical specimen of radius $R$ and height $h$. Determine the temperature distribution in the specimen, assuming that the lateral surface of the cylinder is kept at zero temperature and its ends lose heat in accordance with Newton's law to the surrounding medium at zero temperature.

12. Find the temperature distribution in an infinite circular cylinder $0 \leqslant r \leqslant R$ if its initial temperature is $u_0 = \text{const}$ and a constant heat flux of density $q$ is applied to its surface beginning with time $t = 0$.

13. Determine the natural oscillations (i.e. find the eigenvalues and eigenfunctions) of a circular cylinder of length $h$ subject to the boundary conditions of types I, II, and III, respectively.

14. Determine the natural vibrations of a membrane in the form of a circular sector ($r \leqslant R$, $0 \leqslant \varphi \leqslant \alpha$) for types I, II and III boundary conditions.

15. Find the temperature distribution in an infinite cylindrical sector $0 \leqslant r \leqslant R$, $0 \leqslant \varphi \leqslant \alpha$, whose surface is kept at zero temperature; the initial temperature is arbitrary.

16. Find the temperature distribution in a finite circular cylinder whose surface is kept at zero temperature and where the initial temperature is arbitrary.

17. A circular membrane of radius $R$ is loaded with a point mass $m$ at its centre. Find the eigenvalues $\lambda_n$ of this membrane. Compare these with the eigenvalues for $m = 0$. Consider also the two cases $m$ small and $m$ large.

18. Find the coefficients of the expansion of the function $e^{x/2(t-1/t)}$ in integral powers of $t$.

19. Evaluate $I_{\pm 1/2}(x)$, $K_{\pm 1/2}(x)$, $H^{(1)}_{\pm 1/2}(x)$, $H^{(2)}_{\pm 1/2}(x)$, $N_{\pm 1/2}(x)$.

20. Find the steady-state distribution of the concentration of an unstable gas inside an infinite circular cylinder if the concentration on its surface is maintained at the constant value $u_0$.

21. Solve Problem 20 for the region outside the cylinder.

22. Find the electrostatic field inside the cylinder $0 \leqslant r \leqslant R$, $0 \leqslant z \leqslant h$, whose ends and lateral surface are maintained at potentials $u_1$, $u_2$ and $u_0$, respectively.

23. Find the steady-state temperature distribution in a circular cylinder of height $h$, whose lower face is thermally insulated and whose upper face loses heat in accordance with Newton's law to the medium at zero temperature. The lateral surface is maintained at temperature $u|_{r=R} = f(z)$.

24. The wall of a cylindrical channel drilled through an infinite flat plate of thickness $h$ is maintained at the temperature $u_0 = \text{const}$. Find the steady-state temperature distribution in the plate if its faces are maintained at zero temperature.

25. Find the temperature distribution in a cylinder ($0 \leqslant r \leqslant R$, $0 \leqslant z \leqslant h$) if its initial temperature is zero, and if from time $t = 0$ onwards the base of the cylinder $z = h$ is maintained at the temperature $u_0 = \text{const}$, while the remaining part of the surface is kept at zero temperature.

# 12

# Spherical Harmonics

The simplest class of spherical harmonics is that of the Legendre polynomials $P_n(\cos\theta)$. In this chapter we shall be concerned with their properties.

## 12.1 LEGENDRE POLYNOMIALS

**12.1.1** The simplest and most rapid derivation of these polynomials is to obtain them by means of a generating function.

Let us expand the function $\Psi(x, t) = (1-2xt+t^2)^{-1/2}$ into a power series in terms of $t$. We have

$$\Psi(x, t) = P_0(x)+P_1(x)t+ \ldots +P_n(x)t^n+ \ldots \tag{1}$$

It will be shown below that the coefficients of this expansion, $P_n(x)$, are the Legendre polynomials.

The function $\Psi(x, t)$ is called the generating function for the Legendre polynomials. If we set $x = 1$ in the expansion (1), we obtain

$$\Psi(1, t) = \frac{1}{1-t} = 1+t+ \ldots +t^n+ \ldots$$

Consequently, $P_n(1) = 1$.
Substituting $x = -1$ into (1), we obtain

$$\Psi(-1, t) = \frac{1}{1+t} = 1-t+ \ldots +(-1)^n t^n + \ldots$$

Consequently, $P_n(-1) = (-1)^n$. It is clear that

$$P_n(x) = \frac{1}{n!}\frac{\partial^n \Psi}{\partial t^n}\bigg|_{t=0} \tag{2}$$

On the other hand, the $n$-th order derivative with respect to $t$ of the function $\Psi$ for $t = 0$ is given by

$$\frac{\partial^n \Psi}{\partial t^n}\bigg|_{t=0} = \frac{n!}{2\pi i}\int_C \frac{\Psi(x, \xi)}{\xi^{n+1}}\,\mathrm{d}\xi \tag{2$_1$}$$

where $C$ is a closed contour surrounding the point $\xi = 0$. In the integral given by ($2_1$), let us substitute

$$\sqrt{1-2x\xi+\xi^2} = 1-\xi z$$

We obtain

$$P_n(x) = \frac{n!}{2\pi i}\int_{C_1} \frac{\dfrac{1}{2^n n!}(z^2-1)^n}{(z-x)^{n+1}}\,\mathrm{d}z \tag{3}$$

where $C_1$ is a closed contour surrounding the point $z = x$.

Using the formula for the $n$-th derivative of the Cauchy integral, we obtain

$$P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}[(x^2-1)^n] \tag{4}$$

Therefore, $P_n(x)$ is, in fact, the Legendre polynomial of order $n$.

It follows from Equation (4) that $P_{2k}(z)$ and $P_{2k+1}(z)$ are even and odd functions, respectively. It is clear that

$$P_0(x) \equiv 1, \qquad P_1(x) = x, \qquad P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$$

**12.1.2** To obtain the differential equations for $P_n(x)$ consider the function $w = (x^2-1)^n$. It is clear that

$$w' \equiv 2nx(x^2-1)^{n-1} \equiv \frac{2nxw}{(x^2-1)}$$

or

$$(x^2-1)w' - 2nxw = 0$$

Differentiating the above identity $n+1$ times, we obtain

$$(x^2-1)\,[w^{(n)}]'' + 2x[w^{(n)}]' - n(n+1)w^{(n)} =$$

Therefore, the function $w^{(n)}(x)$ and, consequently, $P_n(x)$ [since $P_n(x) = (1/2^n n!)w^{(n)}(x)$] satisfies the equation

$$(1-x^2)y''-2xy'+\lambda y = 0 \qquad [\lambda = n(n+1)] \tag{5}$$

This is the Legendre differential equation. It can also be written in the form

$$\frac{d}{dx}[(1-x^2)y']+\lambda y = 0 \tag{5$_1$}$$

The second solution of (5), which is linearly independent of $P_n(x)$, has a logarithmic singularity at the point $x = \pm 1$, in accordance with the theorem in Section 11.1.2.

*Remark* The Legendre polynomials can also be generated in another way. One can seek the solution of Equation (5) which is bounded in the range $[-1, 1]$ in the form of the power series $\sum_{k=0}^{\infty} c_k x^k$. For $\lambda = n(n+1)$ this series cuts off at the term of power $n$, i.e. for $\lambda = n(n+1)$ the solution will be a polynomial of order $n$, i.e. $\tilde{P}_n(x)$. It differs from the Legendre polynomial of order $n$ by a constant factor. This factor can be chosen so that $\tilde{P}_n(1) = 1$.

**12.1.3** The Legendre polynomials are orthogonal in the range $[-1, 1]$ with weight $\rho(x) \equiv 1$, i.e.

$$\int_{-1}^{1} P_n(x)P_k(x)\,dx = 0, \qquad \text{if } n \neq k$$

In fact, consider the two identities

$$\frac{d}{dx}[(1-x^2)P_n']+n(n+1)P_n(x) \equiv 0$$

$$\frac{d}{dx}[(1-x^2)P_k']+k(k+1)P_k(x) \equiv 0$$

Let us multiply the first of these by $P_k(x)$ and the second by $P_n(x)$. Let us then subtract one from the other and integrate the result with respect to $x$ within the range $[-1, 1]$. The result is

$$\int_{-1}^{1} \left\{P_k \frac{d}{dx}[(1-x^2)P_n'] - P_n \frac{d}{dx}[(1-x^2)P_k']\right\} dx$$

$$= [k(k+1)-n(n+1)] \int_{-1}^{1} P_n(x)P_k(x)\,dx$$

or

$$\int_{-1}^{1} \frac{d}{dx}\{(1-x^2)(P_n' P_k - P_n P_k')\}\, dx = (k-n)(k+n+1)\int_{-1}^{1} P_n(x)\, P(x)\, dx$$

Consequently,

$$\int_{-1}^{1} P_k(x)\, P_n(x)\, dx = \frac{1}{(k-n)(k+n+1)}\{(1-x^2)(P_n' P_k - P_n P_k')\}_{-1}^{1} = 0$$

when $n \neq k$.

**12.1.4** We shall now establish the validity of the following two recurrence relations:

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + n\, P_{n-1}(x) \equiv 0 \tag{6}$$

$$P_n(x) = \frac{1}{2n+1}[P_{n+1}'(x) - P_{n-1}'(x)] \tag{7}$$

Differentiating the expansion given by (1) with respect to $t$ and $x$ we obtain

$$\frac{\partial \Psi}{\partial t} = \frac{(x-t)\Psi}{1-2xt+t^2} \equiv P_1 + 2P_2 t + \ldots + n\, P_n t^{n-1} + \ldots$$

$$\frac{\partial \Psi}{\partial x} = \frac{t\Psi}{1-2xt+t^2} \equiv P_0' + P_1' t + \ldots + P_n' t^n + \ldots$$

or

$$(x-t)(P_0 + P_1 t + \ldots + P_n t^n + \ldots) \equiv (1-2xt+t^2)(P_1 + 2P_2 t + \ldots + n\, P_n t^{n-1} + \ldots)$$

$$t(P_0 + P_1 t + \ldots + P_n t^n + \ldots) \equiv (1-2xt+t^2)(P_0' + P_1' t + \ldots + P_n' t^n + \ldots)$$

Comparison of the coefficients of equal powers of $t$ in these identities yields

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + n\, P_{n-1}(x) \equiv 0 \tag{6}$$

and

$$P_n(x) \equiv P_{n+1}'(x) - 2x P_n'(x) + P_{n-1}'(x) \tag{8}$$

Differentiating (6), we obtain

$$(n+1)P'_{n+1}(x)-(2n+1)P_n(x)-(2n+1)xP'_n(x)+nP'_{n-1}(x) \equiv 0$$

If we now eliminate $xP'_n(x)$ from this result and from (8), we obtain the identity given by (7).

*Remark 1* Using (6) and the formulae

$$P_0(x) \equiv 1, \qquad P_1(x) = x$$

we can determine all the Legendre polynomials.

*Remark 2* The relationship given by (7) enables us to express the integral of the Legendre polynomial $\int P_n(x)\mathrm{d}x$ in terms of the polynomials $P_{n+1}(x)$ and $P_{n-1}(x)$.

**12.1.5** Let us now evaluate the square of the norm $\|P_n\|^2 = \int\limits_{-1}^{1} P_n^2(x)\mathrm{d}x$. If we express one of the factors in the integrand in terms of $P_{n-1}$ and $P_{n-2}$, in accordance with (6) and replace $n$ by $n-1$, we obtain

$$\|P_n\|^2 = \int\limits_{-1}^{1} P_n P_n \,\mathrm{d}x = \int\limits_{-1}^{1} P_n\left\{\frac{2n-1}{n}xP_{n-1}-\frac{n-1}{n}P_{n-2}\right\}\mathrm{d}x$$

$$= \frac{2n-1}{n}\int\limits_{-1}^{1} xP_n P_{n-1}\,\mathrm{d}x$$

where we have used the orthogonality of the polynomials $P_n$ and $P_{n-2}$. In the last integral the product $xP_n$ can be expressed in terms of $P_{n+1}$ and $P_{n-1}$ in accordance with (6). This yields

$$\|P_n\|^2 = \frac{2n-1}{n}\int\limits_{-1}^{1} P_{n-1}\left\{\frac{n+1}{2n+1}P_{n+1}+\frac{n}{2n+1}P_{n-1}\right\}\mathrm{d}x$$

$$= \frac{2n-1}{2n+1}\int\limits_{-1}^{1} P_{n-1}^2\,\mathrm{d}x$$

or

$$\|P_n\|^2 = \frac{2n-1}{2n+1}\|P_{n-1}\|^2 \tag{9}$$

Here again, we have used the orthogonality of the polynomials $P_{n-1}$ and $P_{n+1}$.

From (9) we have

$$\|P_k\|^2 = \frac{3 \cdot \|P_1\|^2}{2k+1} = \frac{2}{2k+1} \tag{10}$$

and hence $\|P_1\|^2 = 2/3$.

**12.1.6** *Theorem* All the zeros of the Legendre polynomials are simple and lie in the range $(-1, 1)$.

We shall now prove a more general theorem. We shall say that the polynomials $\{q_n(x)\}$ form a normal system if they include the polynomials of all non-negative orders.

*Theorem* If the polynomials $\{q_n(x)\}$ $[q_0(x) \equiv 1]$ are orthogonal in the range $(a, b)$ with weight $\rho(x) > 0$ and form a normal system, all the zeros of the polynomial $q_n(x)$ are simple and lie in the interval $(a, b)$.

*Proof* Since the polynomials $q_n(x)$ are orthogonal, we have (for $n > 0$)

$$\int_a^b 1 \cdot q_n(x)\rho(x)\,\mathrm{d}x = 0$$

Consequently, $q_n(x)$ changes sign $k$ times in the interval $(a, b)$ $(k \geqslant 1)$.

Suppose that this occurs at the point $x_1, x_2, \ldots, x_k$. We then have $q_n(x) = (x-x_1)(x-x_2)\ldots(x-x_k)\varphi_n(x)$, where $\varphi_n(x)$ does not change sign in $(a, b)$. It is clear that to establish the theorem it is sufficient to show that $k = n$.

Let us suppose that $k < n$. We then have the expansion

$$R_k(x) = (x-x_1)(x-x_2)\ldots(x-x_k) = a_0 q_0 + a_1 q_1(x) + \ldots + a_k q_k(x)$$

in which $a_k \neq 0$. It is clear that $\int_a^b q_n(x) R_k(x)\rho(x)\,\mathrm{d}x = 0$ since $q_n$ and $q_r$ are orthogonal $(r = 0, 1, 2, \ldots, k)$. On the other hand,

$$0 = \int_a^b q_n(x) R_k(x)\rho(x)\,\mathrm{d}x = \int_a^b R_k^2(x)\varphi_n(x)\rho(x)\,\mathrm{d}x > 0$$

We have thus reached a contradiction. Consequently, $k = n$. This theorem leads to the theorem on the zeros of the Legendre polynomial. Moreover, the zeros of these polynomials are arranged symmetrically with respect to $x = 0$.

**12.1.7** The Legendre polynomials can be represented by

$$P_n(x) = \frac{1}{2\pi} \int_{-1}^{1} [x + i\sqrt{1-x^2}\sin\varphi]^n \, d\varphi \tag{11}$$

To obtain this integral representation, let us take $C_1$ in Equation (3) to be a circle of radius $R$, $R = \sqrt{1-x^2}$ ($|x| < 1$) centred on

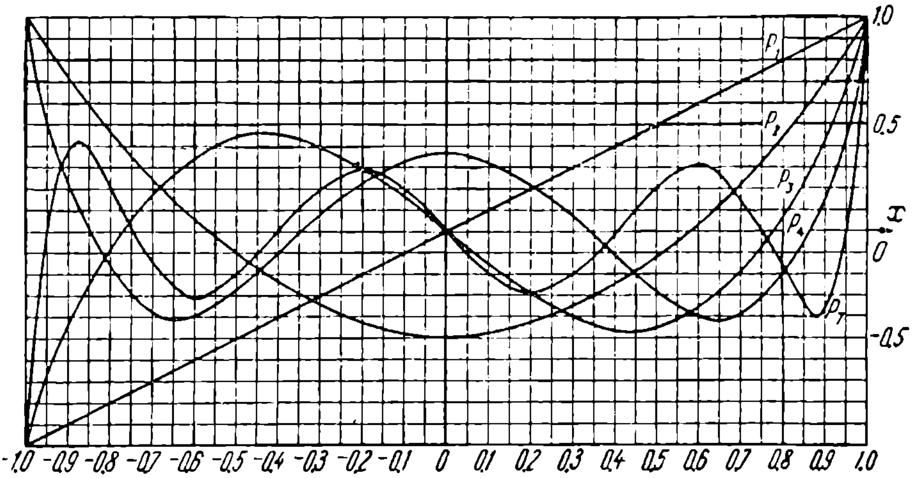



Fig.12.1

the point $z = x$ and then let us substitute $z = x + \sqrt{1-x^2}\,e^{i\varphi}$ so that $dz = i\sqrt{1-x^2}\,e^{i\varphi}\,d\varphi$. We have

$$\begin{aligned} z^2 - 1 &= x^2 - 1 + (1-x^2)e^{2i\varphi} + 2x\sqrt{1-x^2}\,e^{i\varphi} \\ &= \sqrt{1-x^2}\,e^{i\varphi}[2x + \sqrt{1-x^2}\,(e^{i\varphi} - e^{-i\varphi})] \\ &= 2\sqrt{1-x^2}\,e^{i\varphi}[x + i\sqrt{1-x^2}\sin\varphi] \end{aligned}$$

Substituting for $z-x$, $z^2-1$ and $dz$ in (3), we obtain

$$P_n(x) = \frac{1}{2\pi} \int_0^{2\pi} [x + i\sqrt{1-x^2}\sin\varphi]^n \, d\varphi$$

It follows from this that

$$|P_n(x)| < 1 \quad \text{where} \quad x \in (-1, 1) \tag{12}$$

Fig. 12.1 shows graphs of the first few Legendre polynomials.

**12.1.8** *Example 1* Determine the potential inside a hollow sphere of radius $R$ which consists of two hemispheres insulated from each other and charged to potentials $v_1$ and $v_2$.

The mathematical formulation of the problem is as follows. It is required to find the solution $u(r, \theta)$ of the equation $\nabla^2 u = 0$ in the region $0 \leqslant r \leqslant R$ which satisfies the boundary conditions

$$|u(0, \theta)| < \infty, \qquad u(R, \theta) = \begin{cases} v_1, & 0 \leqslant \theta < \dfrac{\pi}{2} \\ v_2, & \dfrac{\pi}{2} < \theta < \pi \end{cases}$$

*Solution* Let us find, to begin with, bounded solutions of $\nabla^2 u = 0$ of the form $u = f(r)\,\psi(\theta)$. Separating the variables, we obtain

$$\frac{\dfrac{d}{dr}(r^2 f')}{f} = \frac{-\dfrac{1}{\sin\theta}\dfrac{d}{d\theta}(\psi' \sin\theta)}{\psi} = \lambda$$

and, consequently,

$$\frac{d}{dr}(r^2 f') - \lambda f = 0, \qquad \frac{1}{\sin\theta}\frac{d}{d\theta}(\psi' \sin\theta) + \lambda\psi = 0$$

In the last equation let us substitute $\xi = \cos\theta$, so that

$$(1-\xi^2)\frac{d^2\psi}{d\xi^2} - 2\xi\frac{d\psi}{d\xi} + \lambda\psi = 0$$

which for $\lambda = n(n+1)$ has a bounded solution in $[-1, 1]$ in the form of a Legendre polynomial $P_n(\xi)$. For such values of $\lambda$ the equation for $f(r)$ has a bounded solution of the form $f(r) = r^n$. The solution of the original problem will now be sought in the form

$$u(r, \theta) = \sum_{n=0}^{\infty} c_n r^n P_n(\cos\theta) \tag{13}$$

The coefficients $c_n$ can be determined from the second boundary condition using the orthogonality of Legendre polynomials:

$$c_n = \frac{2n+1}{2}\int_0^{\pi} u(R, \theta) P_n(\cos\theta)\sin\theta \, d\theta$$

$$= \frac{2n+1}{2}\left\{v_2 \int_{-1}^{0} P_n(\xi)\, d\xi + v_1 \int_0^1 P_n(\xi)\, d\xi\right\}$$

The last two integrals can be evaluated using Equations (7) and (6) of this chapter. The final result is

$$c_n = \frac{v_2 - v_1}{2} \cdot \frac{2n+1}{n} P_{n+1}(0)$$

In this problem we have used the following theorem on the expansion of the function $\varphi(\xi)$ into a Fourier series in terms of the Legendre polynomials:

If a function $\varphi(\xi)$ is piecewise-continuous together with its first-order derivative $\varphi'(\xi)$, then at each point at which $\varphi(\xi)$ is continuous, its Fourier series in terms of the Legendre polynomials converges to this function.

We shall not prove this theorem here.

*Example 2* Expand the plane wave $v = e^{i\lambda z}$ into a series in terms of the Legendre polynomials and Bessel functions.

*Solution* The function $v = e^{i\lambda z} = e^{i\lambda r \cos\theta}$ is a solution of

$$\Delta v + \lambda^2 v = 0$$

Using spherical polar coordinates, this may be written in the form

$$\frac{1}{r^2} \frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}(v_\theta \sin\theta) + \lambda^2 v = 0$$

We shall seek the solution of this equation in the form $v = f(r)\psi(\theta)$. Separating the variables we obtain

$$\frac{1}{r^2} \cdot \frac{d}{dr}(r^2 f') + \left(\lambda^2 - \frac{\mu}{r^2}\right) f = 0$$

$$\frac{1}{\sin\theta} \frac{d}{d\theta}(\psi' \sin\theta) + \mu\psi = 0$$

Bounded solutions of the equation for $\psi$ are obtained when $\mu = n(n+1)$ in the form of the Legendre polynomials

$$\psi(\theta) = P_n(\cos\theta)$$

Substituting

$$f(r) = \frac{\varphi(r)}{\sqrt{r}}$$

we obtain

$$\varphi'' + \frac{1}{r}\varphi' + \left[\lambda^2 - \frac{\left(n+\frac{1}{2}\right)^2}{r^2}\right]\varphi = 0$$

The bounded solution of this equation is the function $J_{n+1/2}(\lambda r)$. Therefore, the equation which the travelling wave under consideration must satisfy has a family of solutions of the form $(1/\sqrt{r})J_{n+1/2}(\lambda r)P_n(\cos\theta)$. It is thus natural to set

$$e^{i\rho\cos\theta} = \sum_{n=0}^{\infty} \frac{c_n}{\sqrt{\rho}} J_{n+1/2}(\rho) P_n(\cos\theta) \tag{14}$$

Using the orthogonality of the Legendre polynomials we find that

$$c_n \frac{J_{n+1/2}(\rho)}{\sqrt{\rho}} = \frac{2n+1}{2} \int_{-1}^{1} e^{i\rho\xi} P_n(\xi)\, d\xi$$

Integrating $n$ times by parts on the right, we obtain

$$\frac{2c_n}{2n+1} \frac{J_{n+1/2}(\rho)}{\sqrt{\rho}} = \frac{1}{i\rho} e^{i\rho\xi} P_n(\xi)\Big|_{-1}^{1} - \frac{1}{(i\rho)^2}[e^{i\rho\xi}(P_n'(\xi)]_{-1}^{1}$$

$$+ \dots + (-1)^n \frac{1}{(i\rho)^n}[e^{i\rho\xi} P_n^{(n)}(\xi)]_{-1}^{1}$$

This result is valid for any $\rho$. For large $\rho$ we can replace the function $J_{n+1/2}(\rho)$ by its asymptotic representation. The result is

$$\frac{2c_n}{2n+1} \frac{\sqrt{2}}{\rho\sqrt{\pi}} \left\{\cos\left[\rho - \left(n+\frac{1}{2}\right)\frac{\pi}{2} - \frac{\pi}{4}\right] + O\left(\frac{1}{\rho}\right)\right\}$$

$$= \frac{2\sqrt{2}c_n}{\sqrt{\pi}(2n+1)} \frac{1}{\rho}\left\{\sin\left(\rho - n\frac{\pi}{2}\right) + O\left(\frac{1}{\rho}\right)\right\}$$

$$= \frac{1}{i\rho} e^{i\rho\xi} P_n(\xi)\Big|_{-1}^{1} + \dots + \frac{(-1)^n}{(i\rho)^n}[e^{i\rho\xi} P_n^{(n)}(\xi)]_{-1}^{1}$$

This shows that

$$\frac{2\sqrt{2}}{\sqrt{\pi}}\,\frac{c_n \sin\left(\rho - n\frac{\pi}{2}\right)}{2n+1} = \frac{1}{i}\,[e^{i\rho} - (-1)^n e^{-i\rho}]$$

or

$$\frac{2\sqrt{2}}{\sqrt{\pi}}\,\frac{c_n \sin\left(\rho - n\frac{\pi}{2}\right)}{2n+1} = \frac{i^n}{i}\,[i^{-n} e^{i\rho} - i^n e^{-i\rho}]$$

$$= \frac{i^n}{i}\left[e^{i\left(\rho - n\frac{\pi}{2}\right)} - e^{-i\left(\rho - n\frac{\pi}{2}\right)}\right]$$

$$= 2 i^n \sin\left(\rho - n\frac{\pi}{2}\right)$$

Hence, we find that

$$c_n = \sqrt{\frac{\pi}{2}}\, i^n (2n+1) = \sqrt{2\pi}\, i^n \left(n + \frac{1}{2}\right) \tag{15}$$

*Example 3* Determine the perturbation of a plane acoustic wave $u_0(M, t)$ due to the presence of a sphere of radius $R$ with perfectly rigid walls, i.e. consider the scattering of sound by a sphere.

We shall suppose that the centre of the sphere lies at the origin of the coordinates. Motion outside the sphere will be described by the function $u(M, t) = u_0(M, t) + v(M, t)$, where $v(M, t)$ is the required perturbation. Since the functions $u(M, t)$ and $u_0(M, t)$ are solutions of the equation

$$a^2 \Delta u = u_{tt}$$

it follows that $v(M, t)$ will also be a solution of this equation. The functions $u$, $u_0$ and $v$ will be interpreted as velocity potentials. On the surface of the sphere we must then have $\left.\frac{\partial u}{\partial n}\right|_{r=R} = 0.$ Therefore, the problem for $v$ can be formulated as follows:

$$a^2 \Delta v = v_{tt} \qquad \text{where} \quad r > R$$

$$\left.\frac{\partial v}{\partial n}\right|_{r=R} = -\left.\frac{\partial u_0}{\partial n}\right|_{r=R}, \quad |v| < \infty$$

It is clear that one can choose a Cartesian set of coordinates so that the plane wave $u_0$ can be written in the form

$$u_0 = e^{ikz} e^{-ikat}$$

We shall seek $v(M, t)$ in the form $v = \Phi(M) e^{-ikat}$. The problem for $\Phi(M)$ can then be formulated as follows:

$$\Delta\Phi + k^2\Phi = 0 \qquad \text{where} \quad r > R$$

$$\left.\frac{\partial \Phi}{\partial n}\right|_{r=R} = -\left.\frac{\partial \varphi_0}{\partial n}\right|_{r=R}, \quad |\Phi| < \infty$$

where $\varphi_0 = e^{ikz}$.

In view of the expansion given by (14), it is natural to seek $\Phi(M)$ in the form of the series

$$\Phi = \sum_{m=0}^{\infty} \Phi_m(M)$$

where $\Phi_m(M)$ is the solution of problem

$$\Delta\Phi_m + k^2\Phi_m = 0 \quad \text{where} \quad r > R \tag{15$_1$}$$

$$\left.\frac{\partial \Phi_m}{\partial n}\right|_{r=R} = -\frac{\partial}{\partial r}\{A_m(r) P_m(\cos\theta)\}_{r=R}, \quad |\Phi_m| < \infty \tag{15$_2$}$$

where $A_m(r)\, P_m(\cos\theta)$ is the term of number $m$ in the expansion (14). If we assume that $\Phi_m = B_m(r)\, \Psi_m(\theta)$, we see that, in view of the boundary condition (15$_2$), the function $\Psi_m(\theta)$ should be equal to $P_m(\cos\theta)$.

Substituting $\Phi_m = B_m(r)\, P_m(\cos\theta)$ into (15$_1$), we obtain

$$B_m'' + \frac{2}{r} B_m' + \left[k^2 - \frac{m(m+1)}{r^2}\right] B_m = 0$$

If we now substitute $B_m = D_m/\sqrt{r}$, we obtain

$$D_m'' + \frac{1}{r} D_m' + \left[k^2 - \frac{\left(m+\frac{1}{2}\right)^2}{r^2}\right] D_m = 0 \tag{15$_3$}$$

This is the differential equation for cylindrical functions with index $\nu = m + 1/2$. A physical interpretation of the problem suggests that the function $v(M, t)$ should be a superposition of spherical diverging waves of the form $(1/r) e^{ik(r-at)}$. Since for large $r$ the

asymptotic behaviour of this kind is exhibited only by the Hankel function

$$H_\nu^{(1)}(kr) = \sqrt{\frac{2}{\pi r}}\, e^{ik\left(r-\nu\cdot\frac{\pi}{2}-\frac{\pi}{4}\right)}\left[1+O\left(\frac{1}{r}\right)\right]$$

the general solution of $(15_3)$ must be written in the form

$$D_m(r) = \alpha_m H^{(1)}_{m+1/2}(kr)+\beta_m H^{(2)}_{m+1/2}(kr)$$

and only the first term need be retained (it represents a converging wave). Therefore, we have

$$B_m(r) = \alpha_m h_m(kr) \quad \text{where} \quad h_\nu(kr) = \frac{1}{\sqrt{kr}} H^{(1)}_{\nu+1/2}(kr)$$

The coefficient $\alpha_m$ can be found from the boundary condition $(15_2)$ which yields

$$B'_m(R) = -A'_m(R)$$

Hence, we find that

$$\alpha_m = \frac{-A'_m(R)}{h'_m(kR)k} = -c_m \frac{j'_m(kR)}{h'_m(kR)} = -\sqrt{2\pi}\, i^m\left(m+\frac{1}{2}\right)\frac{j'_m(kR)}{h'_m(kR)}$$

where $j_m(\rho) = (1/\sqrt{\rho})\, J_{m+1/2}(\rho)$. Consequently,

$$\Phi(M) = \sum_{m=0}^{\infty} \Phi_m(M) = -\sqrt{2\pi}\sum_{m=0}^{\infty}\left(m+\frac{1}{2}\right) i^m \frac{j'_m(kR)}{h'_m(kR)}\, h_m(kr)$$

The functions $P_n(\cos\theta)$ are a sub-class of a much broader class of spherical functions which can be defined with the aid of the associated Legendre functions.

## 12.2 THE ASSOCIATED LEGENDRE FUNCTIONS

**12.2.1** In Section 1 of this chapter we showed that the Legendre polynomials were solutions of Equation (5) for $\lambda = n(n+1)$. In this section we shall consider the more general equation

$$(1-x^2)y''-2xy'+\left(\lambda-\frac{k^2}{1-x^2}\right)y = 0 \tag{16}$$

where $k$ is a non-negative integer.

Solutions of (16) which are bounded in $[-1, 1]$ are called the associated Legendre functions. They can be found by substituting

$$y = (1-x^2)^{k/2} z(x) \tag{17}$$

so that

$$(1-x^2)z''-2x(k+1)z'+[\lambda-k(k+1)]z = 0 \tag{18}$$

or

$$\frac{d}{dx}[(1-x^2)^{k+1}z']+[\lambda-k(k+1)](1-x^2)^k z = 0 \tag{18$_1$}$$

The same expression can be obtained from Equation (5) by differentiating it $k$ times. The solution of (18) and ($18_1$), which is bounded in $[-1, 1]$ for $\lambda = n(n+1)$, is the function $z(x) = \frac{d^k}{dx^k} P_n(x)$. We thus have the following identity:

$$\frac{d}{dx}\left[(1-x^2)^{k+1}\frac{d^{k+1}}{dx^{k+1}}P_n(x)\right]$$

$$\equiv -[n(n+1)-k(k+1)](1-x^2)^k\frac{d^k}{dx^k}P_n(x)$$

$$\equiv (n-k)(n+k+1)(1-x^2)^k\frac{d^k}{dx^k}P_n(x) \tag{19}$$

Consequently, the solution of (16), which is bounded in $[-1, 1]$ for $\lambda = n(n+1)$, i.e. the associated Legendre function is given by

$$P_n^k(x) = (1-x^2)^{k/2}\frac{d^k}{dx^k}P_n(x), \qquad 0 \leqslant k \leqslant n \tag{20}$$

It is clear that $P_n^0(x) = P_n(x)$.

**12.2.2** Henceforth, we shall require only one of the properties of these functions, namely, their orthogonality.

The associated Legendre functions are orthogonal in the range $[-1, 1]$ with weight $\rho(x) \equiv 1$:

$$\int_{-1}^{1} P_n^k(x)P_s^k(x)dx = 0 \quad \text{where} \quad n \neq s \tag{21}$$

*Proof* Substituting

$$A^k_{n,s} = \int_{-1}^{1} P^k_n(x) P^k_s(x) \mathrm{d}x$$

and using Equation (20), we obtain

$$A^k_{n,s} = \int_{-1}^{1} (1-x^2)^k \frac{\mathrm{d}^k}{\mathrm{d}x^k} P_n(x) \frac{\mathrm{d}^k}{\mathrm{d}x^k} P_s(x) \mathrm{d}x$$

$$= (1-x^2)^k \frac{\mathrm{d}^k}{\mathrm{d}x^k} P_n(x) \frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}} P_s(x) \Big|_{-1}^{1}$$

$$- \int_{-1}^{1} \frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}} P_s(x) \frac{\mathrm{d}}{\mathrm{d}x} \left[ (1-x^2)^k \frac{\mathrm{d}^k}{\mathrm{d}x^k} P_n(x) \right] \mathrm{d}x$$

Substitution of the limits of integration in the first term yields zero and, therefore,

$$A^k_{n,s} = - \int_{-1}^{1} \frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}} P_s(x) \frac{\mathrm{d}}{\mathrm{d}x} \left[ (1-x^2)^k \frac{\mathrm{d}^k}{\mathrm{d}x^k} P_n(x) \right] \mathrm{d}x$$

The second factor in the integrand can be transformed with the aid of (19) (replacing $k$ by $k-1$). The final result is

$$A^k_{n,s} = (n-k+1)(n+k) \int_{-1}^{1} \frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}} P_s(x) (1-x^2)^{k-1} \frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}} P_n(x) \mathrm{d}x$$

or

$$A^k_{n,s} = (n-k+1)(n+k) A^{k-1}_{n,s} \tag{22}$$

Using this last expression for $A^{k-1}_{n,s}$, $A^{k-2}_{n,s}$, and so on, we obtain

$$A^k_{n,s} = \frac{(n+k)!}{(n-k)!} A^0_{n,s} \tag{23}$$

Since

$$A^0_{n,s} = \int_{-1}^{1} P_n(x) P_s(x) \mathrm{d}x = 0 \quad \text{if} \quad n \neq s \tag{24}$$

and

$$A_{n,n}^{0} = \int_{-1}^{1} P_n^2(x)\,\mathrm{d}x = \frac{2}{2n+1}$$

it follows that

$$A_{n,s}^{k} = \int_{-1}^{1} P_n^k(x) P_s^k(x)\,\mathrm{d}x = 0 \quad \text{for} \quad n \neq s$$

and

$$A_{n,n}^{k} = \|P_n^k\|^2 = \frac{(n+k)!}{(n-k)!}\,\frac{2}{2n+1}$$

*Remark* It follows from (21) and (22) that the $k$-th order derivatives of the Legendre functions are orthogonal in the range $[-1, 1]$ with the weight $\rho(x) = (1-x^2)^k$.

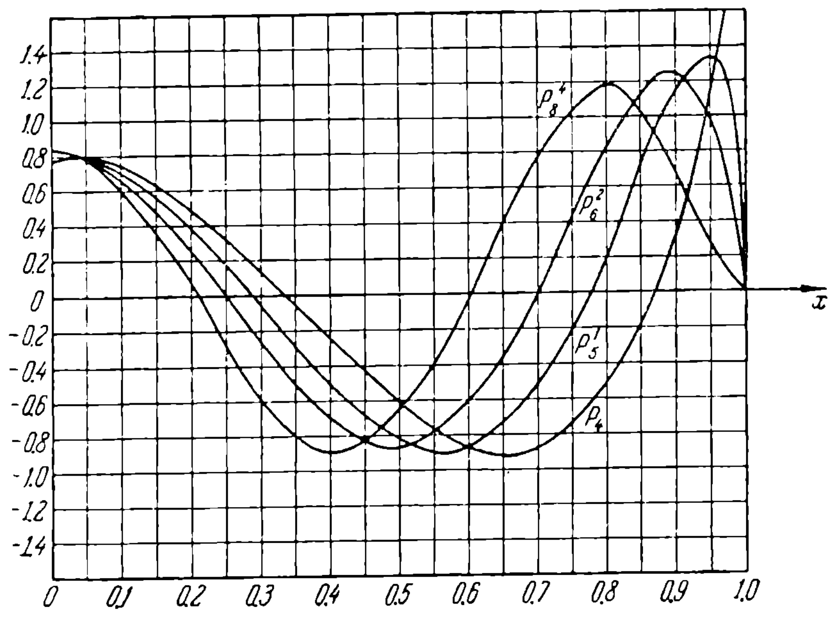



*Fig. 12.2*

Fig. 12.2 shows graphs of the associated Legendre functions.

## 12.3 SURFACE HARMONICS

**12.3.1** We shall seek the solutions of the Laplace equation $\nabla^2 u = 0$, expressed in spherical polar coordinates $r, \theta, \varphi$, in the

form $F(r)$, $Y(\theta, \varphi)$ where the differential equations for $F(r)$, $Y(\theta, \varphi)$ are

$$\frac{d}{dr}(r^2F')-\lambda F = 0 \tag{25}$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2}+\lambda Y = 0 \tag{26}$$

*Definition* The solutions of (26) which are bounded for $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \varphi \leqslant 2\pi$ and are such that $Y(\theta, \varphi+2\pi) \equiv Y(\theta, \varphi)$ are called surface harmonics (or generalised spherical harmonics).

If the bounded solutions of (26) are sought in the form $\Psi(\theta)\Phi(\varphi)$, $\Phi(\varphi+2\pi) \equiv \Phi(\varphi)$, the differential equations for $\Psi(\theta)$ and $\Phi(\varphi)$ are

$$\frac{1}{\sin\theta}\frac{d}{d\theta}(\Psi'\sin\theta)+\left(\lambda-\frac{\mu}{\sin^2\theta}\right)\Psi = 0 \tag{27}$$

$$\Phi''+\mu\Phi = 0 \tag{28}$$

Since the function $\Phi(\varphi)$ is periodic, we find that $\mu = k^2$, where $k$ is an integer and, therefore,

$$\Phi(\varphi) = A\cos k\varphi+B\sin k\varphi$$

If we substitute $\cos\theta = \xi$ into (27), we obtain

$$(1-x^2)\frac{d^2\Psi}{d\xi^2}-2\xi\frac{d\Psi}{d\xi}+\left[\lambda-\frac{k^2}{1-\xi^2}\right]\Psi = 0 \tag{29}$$

which is the same as (16). When $\lambda = n(n+1)$, this equation has the solution $\Psi = P_n^k(\xi)$, which is bounded in $[-1, 1]$. Consequently, surface harmonics of the form $\Psi(\theta)\Phi(\varphi)$, where $\Phi(\varphi+2\pi) \equiv \Phi(\varphi)$ are given by

$$Y_n^k(\theta, \varphi) = P_n^k(\cos\theta)\sin k\varphi$$

and

$$Y_n^{-k}(\theta, \varphi) = P_n^k(\cos\theta)\cos k\varphi \tag{30}$$

$$Y_n^0(\theta, \varphi) = P_n^0(\cos\theta) = P_n(\cos\theta)$$

These functions are called the fundamental surface harmonics of order $n$. It is clear that the functions

$$Y_n(\theta, \varphi) = \sum_{k=-n}^{n} C_k Y_n^k(\theta, \varphi) \tag{31}$$

are also surface harmonics. They are called surface harmonics of order $n$. When $\lambda = n(n+1)$, Equation (25) has solutions of the form

$$F_1(r) = r^n \quad \text{and} \quad F_2(r) = \frac{1}{r^{n+1}}$$

and, consequently,

$$\begin{aligned} u_1(r, \theta, \varphi) &= r^n Y_n(\theta, \varphi) \\ u_2(r, \theta, \varphi) &= \frac{1}{r^{n+1}} Y_n(\theta, \varphi) \end{aligned} \tag{32}$$

are harmonic functions. They are called spherical functions of order $n$.

It follows that surface harmonics of order $n$, $Y_n(\theta, \varphi)$, are the values of spherical functions of order $n$ on a unit sphere.

**12.3.2** Surface harmonics are orthogonal on a unit sphere:

$$\iint_{S_{un}} Y_n(\theta, \varphi) Y_s(\theta, \varphi)\, d\sigma = 0 \quad \text{if} \quad n \neq s$$

or

$$\int_0^{2\pi}\int_0^{\pi} Y_n(\theta, \varphi) Y_s(\theta, \varphi) \sin\theta \, d\theta \, d\varphi = 0 \tag{33}$$

To prove this we note that the fundamental surface harmonics are orthogonal:

$$\int_0^{2\pi}\int_0^{\pi} Y_n^k(\theta, \varphi) Y_s^p(\theta, \varphi) \sin\theta \, d\theta \, d\varphi = 0 \quad \text{for} \quad (n, k) \neq (s, p) \tag{34}$$

since

$$\int_0^{2\pi}\int_0^{\pi} Y_n^k(\theta, \varphi) Y_s^p(\theta, \varphi) \sin\theta \, d\theta \, d\varphi = \int_0^{2\pi} \cos k\varphi \cos p\varphi \, d\varphi \int_{-1}^{1} P_n^k(\xi) P_s^p(\xi) d\xi = 0$$

when $(n, k) \neq (s, p)$ (to be specific, we have assumed that $k > 0$, $p > 0$). If $k \neq p$, the first integral on the right-hand side is equal to zero. If, on the other hand, $k = p$ but $n \neq s$, the second integral will be equal to zero.

The orthogonal property given by (33) follows from the orthogonality of fundamental surface harmonics and from Equation (31).

The square of the norm is given by

$$\|Y_n^k\|^2 = \int_0^{2\pi}\int_0^{\pi} [Y_n^k(\theta, \varphi)]^2 \sin\theta \, d\theta \, d\varphi = \int_0^{2\pi} \cos^2 k\varphi \, d\varphi \int_{-1}^{1} [P_n^k(\xi)]^2 d\xi$$

and, consequently,

$$\|Y_n^k\|^2 = \frac{2\pi}{2n+1}\frac{(n+k)!}{(n-k)!}\varepsilon, \qquad \varepsilon = \begin{cases} 1, & k \neq 0 \\ 2, & k = 0 \end{cases} \tag{34'}$$

**12.3.3** *Theorem* The spherical functions $r^n Y_n(\theta, \varphi)$ are homogeneous harmonic polynomials of order $n$ in the variables $x, y, z$.

*Proof* Since

$$Y_n(\theta, \varphi) = \sum_{k=-n}^{n} C_k Y_n^k(\theta, \varphi)$$

it is sufficient to establish this theorem for the functions $r_n Y_n^k(\theta, \varphi)$. To be specific, we shall assume that $k > 0$. We then have

$$\begin{aligned} Y_n^k(\theta, \varphi) &= P_n^k(\xi) \cos k\varphi = (1-\xi^2)^{k/2} \frac{d^k}{d\xi^k} P_n(\xi) \cos k\varphi \\ &= (1-\xi^2)^{k/2} \frac{d^k}{d\xi^k} \sum_{q=0} a_q \xi^{n-2q} \cos k\varphi \\ &= (1-\xi^2)^{k/2} \sum_{q=0} b_q \xi^{n-2q-k} \cos k\varphi \end{aligned}$$

where $\xi = \cos\theta$. It is clear that it is sufficient to establish the theorem for functions of the form $r^n \sin^k \theta (\cos\theta)^{n-2q-k} \cos k\varphi$. For these functions we have

$$\begin{aligned} r^n \sin^k \theta (\cos\theta)^{n-2q-k} \cos k\varphi &= r^k \sin^k \theta \, \mathrm{Re}(e^{ik\varphi}) r^{2q} r^{n-2q-k} (\cos\theta)^{n-2q-k} \\ &= \mathrm{Re}\,(x+iy)^k (x^2+y^2+z^2)^{2q} z^{n-2q-k} \end{aligned}$$

This is clearly a homogeneous polynomial of order $n$.

*Example* Determine the temperature at the internal points of a sphere of radius $R$, whose surface is maintained at zero temperature; the initial temperature was $f(r, \theta, \varphi)$.

The mathematical formulation of the problem is as follows. It is required to find the solution of $\nabla^2 u = (1/a^2)u_t$ in the range

$0 \leqslant r \leqslant R,\ 0 \leqslant \theta \leqslant \pi,\ 0 \leqslant \varphi \leqslant 2\pi,\ t > 0$, which satisfies the boundary conditions

$$u(R, \theta, \varphi, t) = 0, \qquad |u(0, \theta, \varphi, t)| < \infty$$
$$u(r, \theta, \varphi+2\pi, t) \equiv u(r, \theta, \varphi, t)$$

and the initial conditions $u(r, \theta, \varphi, 0) = f(r, \theta, \varphi)$.

*Solution* We shall find solutions of the equation $\nabla^2 u = (1/a^2)u_t$ in the form $A(r)\,Y(\theta, \varphi)\,B(t)$ which satisfy the boundary conditions uniquely. Separating the variables we find that

$$B' + a^2\alpha B = 0 \tag{35}$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2} + \lambda Y = 0$$
$$|Y(\theta, \varphi)| < \infty, \qquad Y(\theta, \varphi+2\pi) \equiv Y(\theta, \varphi) \tag{36}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}(r^2A') + (\alpha r^2 - \lambda)A = 0, \qquad |A(0)| < \infty, \qquad A(R) = 0 \tag{37}$$

The solutions of (36) for $\lambda = n(n+1)$ are the surface harmonics $Y_n(\theta, \varphi)$. If in the equation for $A(r)$ we substitute $A(r) = z(r)/\sqrt{r}$, we obtain

$$z'' + \frac{1}{r}z' + \left[\alpha - \frac{\left(n+\frac{1}{2}\right)^2}{r^2}\right]z = 0$$

The general solution of this can be written in the form

$$z(r) = M\,J_{n+1/2}(\sqrt{\alpha}\,r) + N\,J_{-n-1/2}(\sqrt{\alpha}\,r)$$

and, consequently,

$$A(r) = M\frac{J_{n+1/2}(\sqrt{\alpha}\,r)}{\sqrt{r}} + N\frac{J_{n-1/2}(\sqrt{\alpha}\,r)}{\sqrt{r}}$$

Since $A(0)$ is bounded, we find that $N = 0$; $M$ can be set equal to unity.

Therefore,

$$A(r) = \frac{1}{\sqrt{r}}J_{n+1/2}(\sqrt{\alpha}\,r)$$

From the condition $A(R) = 0$ we obtain the equation for $\alpha$:

$$J_{n+1/2}(\mu) = 0, \qquad \mu = \sqrt{\alpha}\,R$$

Let

$$\mu_{1,n},\ \mu_{2,n},\ \ldots,\ \mu_{s,n},\ \ldots$$

be the positive roots of this equation. We then have $\alpha_{s,n} = \mu_{s,n}^2/R^2$. Solutions of (37) are of the form

$$A_{n,s}(r) = \frac{1}{\sqrt{r}} J_{n+1/2}\left(\frac{\mu_{s,n}}{R} r\right)$$

Returning now to Equation (35) we find that

$$B_{s,n} = C_{s,n}\, e^{-\frac{a^2\mu_{s,n}^2}{R^2} t}$$

Consequently, the required partial solutions of the original problem which satisfy the boundary condition uniquely are the functions

$$C_{s,n}\, e^{-\frac{a^2\mu_{s,n}^2}{R^2} t} \frac{1}{\sqrt{r}} J_{n+1/2}\left(\frac{\mu_{s,n}}{R} r\right) Y_n(\theta, \varphi)$$

The solution of the original problem can be written in the form

$$u(r, \theta, \varphi, t) = \sum_{n=0}^{\infty}\sum_{s=1}^{\infty}\sum_{k=0}^{n} \frac{1}{\sqrt{r}} J_{n+1/2}\left(\frac{\mu_{s,n}}{R} r\right) [C_{s,n,k} Y_n^k(\theta, \varphi)$$

$$+ D_{s,n,k} Y_n^{-k}(\theta, \varphi)]\, e^{-\frac{a^2\mu_{s,n}^2}{R^2} t}$$

The coefficients $C_{s,n,k}$ and $D_{s,n,k}$ can be determined from the initial conditions using the orthogonality of the functions $Y_n^k(\theta, \varphi)$ and the Bessel functions. The final result is

$$C_{s,n,k} = \frac{(n-k)!(2n+1)}{\pi\varepsilon(n+k)!\,[J'_{n+1/2}(\mu_{s,n})]^2 R^2}$$

$$\times \int_0^R \int_0^{2\pi} \int_0^{\pi} r^{3/2} f(r, \theta, \varphi) J_{n+1/2}\left(\frac{\mu_{s,n}}{R} r\right) Y_n^k(\theta, \varphi) \sin\theta \,dr\, d\theta\, d\varphi$$

$$D_{s,n,k} = \frac{(n-k)!(2n+1)}{\pi\varepsilon R^2(n+k)!\,[J'_{n+1/2}(\mu_{s,n})]^2}$$

$$\times \int_0^R \int_0^{2\pi} \int_0^{\pi} r^{3/2} f(r, \theta, \varphi) J_{n+1/2}\left(\frac{\mu_{s,n}}{R} r\right) Y_n^{-k}(\theta, \varphi) \sin\theta \,dr\, d\theta\, d\varphi$$

## PROBLEMS

1. Calculate $P_n(0)$.

2. Determine whether the $k$-th order derivative of the Legendre polynomials $P_{2n}(x)$, where $k$ is fixed, are orthogonal in the range [0, 1]. If they are, determine the weight.

3. Solve Problem 6 of Chapter 2 for arbitrary initial data, placing the origin of the coordinates at the fixed end of the string.

4. Determine the electrostatic field inside and outside a hollow sphere whose upper half is charged to the potential $V_1$, and lower half to the potential $V_2$.

5. Find the expansion in terms of surface harmonics for the spherical charges induced in a perfectly conducting earthed sphere by point charge $e$ placed (a) inside the sphere and (b) outside the sphere.

6. Consider the polarisation of a dielectric sphere of radius $R$ placed in the field of a point charge if the permittivity is $\varepsilon = \varepsilon_1$ for $r < R$ and $\varepsilon = \varepsilon_2$ for $r > R$.

7. Find the potential of a simple layer in the form of a circular disc.

8. Calculate the potential at all points of a conducting sphere of conductivity $\sigma$ when a current $I$ enters the sphere at one of its poles ($\theta = 0$) and leaves at the pole $\theta = \pi$.

9. A sphere loses heat through its surface to the surrounding medium which is maintained at zero temperature. A point heat source of strength $Q$ is placed at an internal point of the sphere. Find the steady-state distribution of temperature inside the sphere.

10. Determine the potential due to a point charge placed between two conducting earthed concentric spheres $r = R_1$ and $r = R_2$, also determine the density of surface charges.

11. Find the steady-state distribution of temperature in a sphere of radius $R$, a part of whose surface $S_1$ ($\theta \leqslant \alpha$) is at a temperature $u_0 =$ const, while the remainder of the surface $S_2$ is maintained at zero temperature.

12. A sphere of radius $R$ is heated by a plane-parallel beam of heat of density $q$ incident on its surface. The sphere loses heat to the surrounding medium maintained at zero temperature in accordance with Newton's law of cooling. Find the steady-state distribution of temperature.

13. Consider the oscillations of a gas in a spherical container due to small oscillations of the wall, assuming that the oscillations

began at $t = 0$. The displacement of the wall is radial and given by $P_n(\cos\theta) f(t)$, where $f(0) = f'(0) = 0$.

14. Find the natural oscillations of a sphere subject to types I, II and III boundary conditions, respectively.

15. Consider the cooling of a sphere of radius $R$, whose surface is maintained at zero temperature. The initial temperature was $f(r, \theta, \varphi)$.

# 13

# Chebyshev-Hermite and Chebyshev-Laguerre Polynomials

## 13.1 CHEBYSHEV–HERMITE POLYNOMIALS

We shall now determine two new classes of orthogonal polynomials which have many applications. There are a number of ways in which they can be defined; we shall derive them by the method of generating functions, which provides the quickest route to the main properties of these polynomials.

**13.1.1** As the generating function we shall take $H(x, t) = e^{2xt-t^2}$ and expand it into a power series in terms of $t$:

$$H(x, t) = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} \tag{1}$$

It will be shown below that the expansion coefficients $H_n(x)$ are the Chebyshev–Hermite polynomials. It is clear that

$$H_n(x) = \left.\frac{\partial^n H(x, t)}{\partial t^n}\right|_{t=0}$$

On the other hand, the $n$-th order derivative $\partial^n H/\partial t^n$ for $t = 0$ is given by

$$\left.\frac{\partial^n H}{\partial t^n}\right|_{t=0} = \frac{n!}{2\pi i} \int_C \frac{e^{2xt-t^2}}{t^{n+1}} \mathrm{d}t$$

where the closed contour $C$ surrounds the point $t = 0$, or

$$H_n(x) = e^{x^2}\frac{n!}{2\pi i}\int\limits_C \frac{e^{-(x-t)^2}}{t^{n+1}}\,dt$$

If we substitute $x-t = \xi$ in the last integral, we obtain

$$H_n(x) = e^{x^2}\frac{n!}{2\pi i}(-1)^n\int\limits_{C_1} \frac{e^{-\xi^2}}{(\xi-x)^{n+1}}\,d\xi$$

where the contour $C_1$ surrounds the point $\xi = x$. Using the formula for the $n$-th derivative of the Cauchy integral, we obtain

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}(e^{-x^2}) \tag{2}$$

It follows from this expression that $H_n(x)$ is a polynomial of order $n$. $H_{2k}(x)$ and $H_{2k+1}(x)$ are even and odd functions, respectively. It is clear that $H_0(x) \equiv 1$, $H_1(x) = 2x$, $H_2(x) = 4x^2-2$, and so on.

**13.1.2** We shall show that $H_n(x)$ is a solution of the equation

$$y''-2xy'+\lambda y = 0 \quad \text{where} \quad \lambda = 2n \tag{3}$$

In fact, differentiating the function $w = e^{-x^2}$ once, we find that $w'+2xw \equiv 0$. Differentiating this identity $n+1$ times, we obtain

$$[w^{(n)}]''+2x[w^{(n)}]'+2nw^{(n)} \equiv 0 \tag{4}$$

Substituting (2) into this identity, we have

$$w^{(n)} = (-1)^n H_n(x)\,e^{-x^2}$$

$$H_n''(x)-2x\,H_n'(x)+2n\,H_n(x) \equiv 0$$

and so on. Equation (3) can be written in the form

$$\frac{d}{dx}(e^{-x^2}y')+\lambda e^{-x^2}y = 0 \tag{5}$$

**13.1.3** *Theorem* The Chebyshev–Hermite polynomials are orthogonal in the range $(-\infty, \infty)$ with weight $\rho(x) = e^{-x^2}$, i.e.

$$\int\limits_{-\infty}^{\infty} H_n(x)H_p(x)e^{-x^2}dx = 0 \quad \text{if} \quad n \neq p \tag{6}$$

*Proof* Consider the two identities

$$\frac{d}{dx}[e^{-x^2}H_n'(x)]+2n\,e^{-x^2}H_n(x) \equiv 0$$

$$\frac{d}{dx}[e^{-x^2}H_p'(x)]+2p\,e^{-x^2}H_p(x) \equiv 0$$

Multiply the first by $H_p(x)$ and the second by $H_n(x)$, subtract one from the other and integrate the difference with respect to $x$ between $-\infty$ and $+\infty$. The result is

$$\int_{-\infty}^{\infty}\left\{H_p\frac{d}{dx}(e^{-x^2}H_n')-H_n\frac{d}{dx}(e^{-x^2}H_p')\right\}dx$$

$$= 2(p-n)\int_{-\infty}^{\infty} H_n(x)H_p(x)\,e^{-x^2}\,dx$$

The left-hand side of this equation can clearly be written in the form

$$\int_{-\infty}^{\infty}\frac{d}{dx}\{(H_pH_n'-H_nH_p')e^{-x^2}\}\,dx$$

and, consequently,

$$2(p-n)\int_{-\infty}^{\infty} H_nH_p\,e^{-x^2}dx = (H_pH_n'-H_nH_p')\,e^{-x^2}\Big|_{-\infty}^{\infty} = 0$$

Since $p \neq n$, this leads directly to Equation (6).

Let us now find the norm $\|H_n\|$. To begin with, let us establish the following recurrence relationships:

$$H_{n+1}(x)-2xH_n(x)+2nH_{n-1}(x) \equiv 0 \tag{7}$$

$$H_n'(x) \equiv 2nH_{n-1}(x) \tag{8}$$

To do this we must find the relationship between the generating function $H(x, t)$ and its partial derivatives $\partial H/\partial t$ and $\partial H/\partial x$. Direct calculation shows that

$$\frac{\partial H}{\partial t} \equiv 2(x-t)H \qquad \text{and} \qquad \frac{\partial H}{\partial x} \equiv 2tH$$

Substituting (1) into these identities, we obtain

$$\sum_{n=1}^{\infty} H_n(x)\frac{t^{n-1}}{(n-1)!} \equiv 2(x-t)\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} \tag{9}$$

$$\sum_{n=0}^{\infty} H_n'(x)\frac{t^n}{n!} \equiv 2t\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} \tag{10}$$

Equating the coefficients of equal powers of $t$ in (9) and (10), we obtain the recurrence relations (7) and (8), respectively.

The identity given by (8) can be used to evaluate the integral

$$\int H_n(x)\,\mathrm{d}x = \frac{1}{2(n+1)} H_{n+1}(x)$$

The identity (7) can be used to evaluate the square of the norm $\|H_n\|^2 = \int\limits_{-\infty}^{\infty} \mathrm{e}^{-x^2} H_n^2\,\mathrm{d}x$:

$$\|H_n\|^2 = \int\limits_{-\infty}^{\infty} \mathrm{e}^{-x^2} H_n^2(x)\,\mathrm{d}x = \int\limits_{-\infty}^{\infty} H_n(x) H_n(x)\mathrm{e}^{-x^2}\,\mathrm{d}x$$

Let us express one of the factors $H_n(x)$ in the integrand by means of (7) in terms of $H_{n-1}$ and $H_{n-2}$ by replacing $n$ by $n-1$. The result is

$$\|H_n\|^2 = \int\limits_{-\infty}^{\infty} \mathrm{e}^{-x^2} H_n(x)\left\{2x H_{n-1}(x) - 2(n-1) H_{n-2}(x)\right\}\mathrm{d}x$$

$$= \int\limits_{-\infty}^{\infty} \mathrm{e}^{-x^2} H_{n-1}(x) 2x H_n(x)\,\mathrm{d}x$$

In this expression we have used the orthogonality of the polynomials $H_{n-2}$ and $H_n$. If we now express $2xH_n(x)$ in terms of $H_{n-1}(x)$ and $H_{n+1}(x)$ through Equation (7), we obtain

$$\|H_n\|^2 = \int\limits_{-\infty}^{\infty} \mathrm{e}^{-x^2} H_{n-1}(x)\left\{H_{n+1}(x) + 2n H_{n-1}(x)\right\}\mathrm{d}x$$

$$= 2n \int\limits_{-\infty}^{\infty} \mathrm{e}^{-x^2} H_{n-1}^2(x)\,\mathrm{d}x$$

or

$$\|H_n\|^2 = 2n\|H_{n-1}\|^2 \tag{11}$$

In this we have used the orthogonality of the polynomials $H_{n-1}$ and $H_{n+1}$. From (11) it follows that

$$\|H_n\|^2 = 2^{n-1}n!\,\|H_1\|^2 = 2^n n! \int_{-\infty}^{\infty} 2x^2 e^{-x^2} dx = 2^n n! \sqrt{\pi} \quad (11_1)$$

Therefore,

$$\|H_n\|^2 = 2^n n! \sqrt{\pi} \tag{12}$$

**13.1.4** The Chebyshev–Hermite polynomials form a normal system. Consequently the theorem of Section 12.6 applies to the polynomials. All the zeros of the polynomials $H_n(x)$ are simple and real.

**13.1.5** The Chebyshev–Hermite functions

$$\psi_n(x) = \frac{H_n(x)}{\|H_n\|} e^{-x^2/2} \tag{13}$$

are frequently used. They vanish at infinity. These functions form an orthogonal system with the weight $\rho(x) = 1$:

$$\int_{-\infty}^{\infty} \psi_n(x)\psi_p(x) dx = 0 \quad \text{if} \quad n \neq p \tag{14}$$

$$\|\psi_n\| = 1 \tag{15}$$

From Equation (5) for the polynomials $H_n(x)$ one can readily obtain the differential equation for the functions $\psi_n(x)$:

$$\psi'' + (\lambda - x^2)\psi = 0 \qquad (\lambda = 2n+1) \tag{16}$$

*Example* Determine the values of $E$ for which the Schrödinger equation for the linear harmonic oscillator

$$\psi'' + \left\{\frac{2mE}{\hbar^2} - \frac{m^2\omega_0^2}{\hbar^2} x^2\right\}\psi = 0 \tag{17}$$

has a solution which is bounded in the range $-\infty < x < \infty$. In this equation $m$, $\omega_0$, $E$ are the mass, the natural frequency and the total energy of the oscillator, respectively, and $\hbar$ is Planck's constant divided by $2\pi$.

If we substitute $z = \sqrt{\omega_0 m/\hbar}\, x$ into (17) this equation reduces to the form given by (16), in which $\lambda = 2E/\omega_0\hbar$:

$$\frac{d^2\psi}{dz^2} + \left(\frac{2E}{\omega_0\hbar} - z^2\right)\psi = 0 \tag{18}$$

When $2E/\omega_0\hbar = 2n+1$, where $n$ is an integer, Equation (17) has the solution $\psi_n(z) = \psi_n(\sqrt{\omega_0 m/\hbar}\, x)$, which is bounded in the range $-\infty < z < \infty$. The Chebyshev–Hermite functions provide solutions of the Laplace equation $\nabla^2 u = 0$ on separation of variables in terms of parabolic coordinates. In fact, if we introduce the parabolic coordinates $\alpha$, $\beta$, $z$, which are related to the Cartesian coordinates $x, y, z$ by

$$x = \frac{c}{2}(\alpha^2-\beta^2), \quad y = c\alpha\beta, \quad z = z \tag{19}$$

(where $c$ is a dimensional factor, $-\infty < \alpha < \infty$, $0 \leqslant \beta < \infty$, $-\infty < z < \infty$), then $\nabla^2 u = 0$ takes the form

$$\nabla^2 u = \frac{1}{c^2(\alpha^2+\beta^2)}\left\{\frac{\partial^2 u}{\partial\alpha^2} + \frac{\partial^2 u}{\partial\beta^2} + c^2(\alpha^2+\beta^2)\frac{\partial^2 u}{\partial z^2}\right\} = 0 \tag{20}$$

We shall seek the solution of this equation in the form $u = A(\alpha)\,B(\beta)\,D(z)$. Separating the variables we obtain the following differential equations:

$$A'' + (\mu - \lambda^2 c^2\alpha^2)A = 0 \tag{21}$$

$$B'' - (\mu + \lambda^2 c^2\beta^2)B = 0 \tag{22}$$

$$D'' + \lambda^2 D = 0$$

where $\lambda^2$ and $\mu$ are unknown parameters. Substituting $\xi = \sqrt{\lambda c}\,\alpha$ and $\eta = i\sqrt{\lambda c}\,\beta$ into (21) and (22), we obtain

$$\frac{d^2 A}{d\xi^2} + \left(\frac{\mu}{\lambda c} - \xi^2\right)A = 0 \tag{23}$$

and

$$\frac{d^2 B}{d\eta^2} + \left(\frac{\mu}{\lambda c} - \eta^2\right)B = 0$$

which is the same as Equation (17).

## 13.2 CHEBYSHEV–LAGUERRE POLYNOMIALS

**13.2.1** As indicated in Section 12.1, we shall derive the Chebyshev–Laguerre polynomials with the aid of a generating function. We can take this to be

$$L^\alpha(x, t) = \frac{1}{(1-t)^{\alpha+1}}\, e^{\frac{-xt}{1-t}}, \qquad \alpha > -1$$

and expand it into a power series in terms of $t$:

$$L^{\alpha}(x, t) = \sum_{n=0}^{\infty} L_n^{\alpha}(x) t^n \tag{24}$$

It will be shown below that the coefficients of the expansion are the Chebyshev–Laguerre polynomials. These are occasionally called the generalised Chebyshev–Laguerre polynomials, whilst the polynomials $L_n^0(x) \equiv L_n(x)$ are called the Chebyshev–Laguerre polynomials. It is clear that

$$L_n^{\alpha}(x) = \frac{1}{n!} \frac{\partial^n L^{\alpha}(x, t)}{\partial t^n}\bigg|_{t=0} = \frac{1}{2\pi i} \int\limits_C \frac{L^{\alpha}(x, t)}{t^{n+1}} \, dt$$

where $C$ is a closed contour surrounding the point $t = 0$. Let us substitute $t = 1 - x/z$ in this integral. We obtain

$$L_n^{\alpha}(x) = x^{-\alpha} e^x \frac{1}{2\pi i} \int\limits_{C_1} \frac{z^{n+\alpha} e^{-x}}{(z-x)^{n+1}} \, dz = \frac{1}{n!} x^{-\alpha} e^x \frac{d^n}{dx^n} (x^{n+\alpha} e^{-x})$$

The contour $C_1$ surrounds the point $z = x$. We can then use the formula for the derivative of the Cauchy integral. Therefore,

$$L_n^{\alpha}(x) = \frac{1}{n!} x^{-\alpha} e^x \frac{d^n}{dx^n} (x^{n+\alpha} e^{-x}) \tag{25}$$

It follows from this formula that $L_n^{\alpha}(x)$ is, in fact, a polynomial of order $n$. It is clear that $L_0^{\alpha}(x) \equiv 1$, $L_n^{\alpha}(x) = 1+\alpha-x$.

**13.2.2** We shall show that the polynomial $L_n^{\alpha}(x)$ is a solution of the equation

$$xy'' + (\alpha+1-x)y' + \lambda y = 0 \tag{26}$$

or

$$\frac{d}{dx}(x^{\alpha+1} e^{-x} y') + \lambda x^{\alpha} e^{-x} y = 0 \quad \text{for} \quad \lambda = n \tag{$26_1$}$$

In fact, differentiating the function $w = x^{n+\alpha} e^{-x}$ once, we obtain

$$w' = (n+\alpha) x^{n+\alpha-1} e^{-x} - x^{n+\alpha} e^{-x}$$

and this yields the identity

$$xw' - (n+\alpha-x)w \equiv 0$$

Differentiating this $n+1$ times, we obtain

$$x[w^{(n)}]''+(x+1-\alpha)[w^{(n)}]'+(n+1)w^{(n)} \equiv 0$$

Using (25) we have

$$w^{(n)} = x^{\alpha} e^{-x} L_n^{\alpha}(x) n!$$

and substituting this for $w^{(n)}$ we obtain the identity

$$x(L_n^{\alpha})''+(\alpha+1-x)(L_n^{\alpha})'+n L_n^{\alpha} \equiv 0$$

We shall now consider some of the properties of the polynomials $L_n^{\alpha}(x)$.

**13.2.3** *Theorem* The Chebyshev–Laguerre polynomials are orthogonal within the range $(0, \infty)$ with weight $\rho(x) = x^{\alpha} e^{-x}$:

$$\int_0^{\infty} L_n^{\alpha}(x) L_p^{\alpha}(x) x^{\alpha} e^{-x} dx = 0 \quad \text{if} \quad n \neq p \quad \text{and} \quad \alpha > -1 \qquad (27)$$

*Proof* Consider the two identities

$$\frac{d}{dx}\left[x^{\alpha+1} e^{-x} \frac{dL_n^{\alpha}(x)}{dx}\right] + n x^{\alpha} e^{-x} L_n^{\alpha}(x) \equiv 0$$

$$\frac{d}{dx}\left[x^{\alpha+1} e^{-x} \frac{dL_p^{\alpha}(x)}{dx}\right] + p x^{\alpha} e^{-x} L_p^{\alpha}(x) \equiv 0$$

Let us multiply the first of these by $L_p^{\alpha}(x)$ and the second by $L_n^{\alpha}(x)$, subtract one from the other and then integrate with respect to $x$ within the range $(0, \infty)$. The result is

$$\int_0^{\infty} \left\{L_p^{\alpha} \frac{d}{dx}\left[x^{\alpha+1} e^{-x} \frac{dL_n^{\alpha}}{dx}\right] - L_n^{\alpha} \frac{d}{dx}\left[x^{\alpha+1} e^{-x} \frac{dL_n^{\alpha}}{dx}\right]\right\} dx$$

$$= (p-n) \int_0^{\infty} L_n^{\alpha}(x) L_p^{\alpha}(x) x^{\alpha} e^{-x} dx$$

The left-hand side of this equation can clearly be written in the form

$$\int_0^{\infty} \frac{d}{dx}\{x^{\alpha+1} e^{-x}[L_p^{\alpha}(L_n^{\alpha})' - (L_p^{\alpha})' L_n^{\alpha}]\} dx$$

Consequently,

$$\int_0^\infty L_n^\alpha(x)\, L_p^\alpha(x)\, x^\alpha\, \mathrm{e}^{-x}\,\mathrm{d}x = \frac{1}{p-n}\{x^{\alpha+1}\,\mathrm{e}^{-x}[(L_n^\alpha)' L_p^\alpha - (L_p^\alpha)' L_n^\alpha]\}_0^\infty = 0$$

When $x = 0$ the integrated part vanishes because of the presence of $x^{\alpha+1} (\alpha > -1)$, and when $x = \infty$, because of the presence of $\mathrm{e}^{-x}$. Since $p \neq n$, Equation (27) follows at once.

**13.2.4** We shall now determine the norm $||L_n^\alpha||$. First we must establish the following two recurrence relations:

$$(n+1)\,L_{n+1}^\alpha(x) - (2n+1+\alpha-x)\,L_n^\alpha(x) + (n+\alpha)\,L_{n-1}^\alpha(x) \equiv 0 \tag{28}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} L_n^\alpha(x) \equiv -L_{n-1}^{\alpha+1}(x) \tag{29}$$

To do this, we must find the connection between the derivative of the function and its partial derivatives, $\partial L^\alpha/\partial x$ and $\partial L^\alpha/\partial t$. Direct calculation shows that

$$(1-2t+t^2)\frac{\partial L^\alpha(x, t)}{\partial t} \equiv [\alpha+1-x-(\alpha+1)t]\,L^\alpha(x, t)$$

and

$$\frac{\partial L^\alpha(x, t)}{\partial x} \equiv -tL^{\alpha+1}(x, t)$$

Substituting (24) into these identities, we obtain

$$(1-2t+t^2)\sum_{n=1}^{\infty} n\,L_n^\alpha(x)\,t^{n-1} \equiv [\alpha+1-x-(\alpha+1)\,t]\sum_{n=0}^{\infty} L_n^\alpha(x)\,t^n \tag{30}$$

and

$$\sum_{n=0}^{\infty} t^n \frac{\mathrm{d}L_n^\alpha(x)}{\mathrm{d}x} \equiv -\,t\sum_{n=0}^{\infty} t^n L_n^{\alpha+1}(x) \tag{31}$$

Equating the coefficients of equal powers of $t$ in (30) and (31), we obtain Equations (28) and (29), respectively.

From (29) it follows that

$$\int L_n^\alpha(x)\,\mathrm{d}x = -L_{n+1}^{\alpha-1}(x) \tag{32}$$

We shall use (28) to evaluate $||L_n^\alpha||^2$:

$$||L_n^\alpha||^2 = \int_0^\infty x^\alpha e^{-x}[L_n^\alpha(x)]^2 dx = \int_0^\infty x^\alpha e^{-x} L_n^\alpha(x) L_n^\alpha(x) dx$$

Using (28) and replacing $n$ by $n-1$, we obtain

$$||L_n^\alpha||^2 = \int_0^\infty x^\alpha e^{-x} L_n^\alpha(x) \{(2n-1+\alpha-x) L_{n-1}^\alpha(x)$$

$$-(n-1+\alpha) L_{n-2}^\alpha(x)\} \frac{1}{n} dx$$

$$= \frac{1}{n} \int_0^\infty x^\alpha e^{-x} L_{n-1}^\alpha(x) [-x L_n^\alpha(x)] dx$$

We have used the orthogonality of the polynomials $L_n^\alpha$ and $L_{n-2}^\alpha$ and the orthogonality of $L_n^\alpha$ and $L_{n-1}^\alpha$. If we use (28) again, we obtain

$$||L_n^\alpha||^2 = \frac{1}{n} \int_0^\infty x^\alpha e^{-x} L_{n-1}^\alpha(x) \{(n+1) L_{n+1}^\alpha(x)$$

$$-(2n+1+\alpha) L_n^\alpha(x) + (n+\alpha) L_{n-1}^\alpha(x)\} dx$$

$$= \frac{n+\alpha}{n} \int_0^\infty x^\alpha e^{-x} [L_{n-1}^\alpha(x)]^2 dx = \frac{n+\alpha}{n} ||L_{n-1}^\alpha||^2$$

or

$$||L_n^\alpha||^2 = \frac{n+\alpha}{n} ||L_{n-1}^\alpha||^2 \tag{33}$$

In these expressions we have used the orthogonality of the polynomials

$$L_{n-1}^\alpha \quad \text{and} \quad L_{n+1}^\alpha, \qquad L_{n-1}^\alpha \quad \text{and} \quad L_n^\alpha$$

From Equation (33) it follows that

$$||L_n^\alpha||^2 = \frac{(n+\alpha)(n+\alpha-1)\ldots(\alpha+2)}{n!} ||L_1^\alpha||^2$$

$$= \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+2)} \int_0^\infty (1+\alpha-x)^2 x^\alpha e^{-x} dx$$

$$= \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+2)} \Gamma(\alpha+2) = \frac{\Gamma(n+\alpha+1)}{n!}$$

Therefore,

$$\|L_n^\alpha\|^2 = \frac{\Gamma(n+\alpha+1)}{n!} \tag{34}$$

It is clear that the Chebyshev polynomials form a normal set. Consequently, the theorem of Section 12.1.6 is valid for them. All the zeros of the polynomials $L_n^\alpha(x)$ are therefore simple and real and lie in the interval $(0, \infty)$.

**13.2.5** The functions

$$\Phi_n^\alpha(x) = \frac{L_n^\alpha(x)}{\|L_n^\alpha\|} x^{\alpha/2} e^{-x/2} \tag{35}$$

are frequently used in applications. They vanish at infinity ($x = +\infty$). These functions have the following property:

$$\int_0^\infty \Phi_n^\alpha(x)\Phi_p^\alpha(x)\,dx = 0 \qquad \text{if } n \neq p \text{ and } \alpha > -1 \tag{36}$$

From Equation ($26_1$) it follows at once that the functions $\Phi_n^\alpha(x)$ are the solutions of the equation

$$\frac{d}{dx}(xy') + \left(\lambda - \frac{x}{4} - \frac{\alpha^2}{4x}\right)y = 0 \tag{37}$$

where

$$\lambda = n + \frac{\alpha+1}{2}$$

The Chebyshev–Laguerre polynomials are encountered in the solution of problems on the propagation of electromagnetic waves in long lines and in the analysis of the motion of electrons in the Coulomb field, as well as in certain other problems.

*Example* Expand the function $f(x) = e^{-x}$ into a Fourier series in terms of the Chebyshev–Laguerre polynomials.

*Solution* In the required expansion

$$e^{-x} = \sum_{n=0}^{\infty} c_n L_n^\alpha(x)$$

the coefficients $c_n$ are given by

$$c_n = \frac{1}{\|L_n^\alpha\|^2} \int_0^\infty x^\alpha \, e^{-2x} L_n^\alpha(x) \, dx = \frac{1}{\|L_n^\alpha\|^2} \int_0^\infty e^{-x} \frac{d^n}{dx^n} (x^{n+\alpha} \, e^{-x}) \, dx$$

Integrating $n$ times by parts, we obtain

$$c_n = \frac{n!}{\Gamma(n+\alpha+1)} \left\{ e^{-x} \frac{d^{n-1}}{dx^{n-1}} (x^{n+\alpha} \, e^{-x}) \Big|_0^\infty \right.$$

$$\left. + e^{-x} \frac{d^{n-2}}{dx^{n-2}} (x^{n+\alpha} \, e^{-x}) \Big|_0^\infty + \ldots + \int_0^\infty x^{n+\alpha} \, e^{-2x} \, dx \right\}$$

Substitution of the limits yields zero.
Since

$$\int_0^\infty x^{n+\alpha} \, e^{-2x} \, dx = \frac{1}{2^{n+\alpha+1}} \Gamma(n+\alpha+1)$$

it follows that

$$c_n = \frac{n!}{2^{n+\alpha+1}}$$

# Appendix

# Definition of Generalised Functions. The δ Function

**A.1** Generalised functions will be introduced by a method similar to that used when real numbers are introduccd with the aid of sequences of rational numbers. Real numbers are introduced so that operations such as the extraction of the root or the taking of logarithms can be carried out. The introduction of generalised functions ensures that the operation of differentiation can always be performed.

The sequence of rational numbers $\{a_n\}$ is called fundamental if for any rational $\varepsilon > 0$ there exists $n_0$, such that for all $n$ and $m > m_0$

$$|a_n - a_m| < \varepsilon$$

The fundamental sequences $\{a_n\}$ and $\{b_n\}$ are called equivalent if

$$\lim_{n\to\infty} |a_n - b_n| = 0$$

Equivalent sequences define a real number.

**A.2** Consider continuous functions defined in the interval $(A, B)$, $-\infty \leqslant A < B \leqslant \infty$. The sequence of functions $\{f_n(x)\}$ which are continuous in $(A, B)$ is called fundamental in $(A, B)$ if there exists an integer $k \geqslant 0$ and another sequencc of functions $\{F_n(x)\}$ which are continuous in $(A\ B)$ which are such that the following two properties are satisfied:

1. $F_n^{(k)}(x) = f_n(x)$;
2. the sequence $\{F_n(x)\}$ converges uniformly in any segment $[\alpha, \beta] \subset (A, B)$ $(F_n(x) \rightrightarrows)$.

The following theorem follows from the definition of a fundamental sequence.

*Theorem 1* If a sequence of functions $\{f_n(x)\}$ which are continuous in $(A, B)$ converges uniformly in any segment $[\alpha, \beta] \subset (A, B)$, then it is a fundamental sequence. In point of fact, we have $F_n(x) \equiv f_n(x)$ and $k = 0$.

*Theorem 2* If $\{f_n(x)\}$ is a fundamental sequence of functions having continuous derivatives of order $m$, $f_n^{(m)}(x)$, the sequence $\{f_n^{(m)}(x)\}$ is also fundamental.

*Proof* For $\{f_n(x)\}$ there exists a number $k \geqslant 0$, and a sequence $\{F_n(x)\}$ which has Properties 1 and 2. For $\{f_n^{(m)}(x)\}$ we can take $k+m$ instead of $k$ and the same sequence $\{F_n(x)\}$. These will clearly satisfy Properties 1 and 2. Consequently, $\{f_n^{(m)}(x)\}$ is a fundamental sequence.

A sequence of functions $\{f_n(x)\}$ is said to be uniformly bounded in $(A, B)$ if there exists a number $M$ such that $|f_n(x)| \leqslant M$ for all $n$ and for all $x \in (A, B)$.

*Theorem 3* If a sequence of functions which are continuous in $(A, B)$ is uniformly bounded in $(A, B)$, and converges uniformly in any segment $[\alpha, \beta] \subset (A, x_0)$ and in any segment $[\alpha, \beta] \subset (x_0, B)$, then it is fundamental in $(A, B)$ where $x_0 \in (A, B)$.

*Proof* Let us take $F_n(x) = \int\limits_{x_0}^{x} f_n(t)\mathrm{d}t$ and $k = 1$. Property 1 will then be satisfied. It remains to show that Property 2 is satisfied. Let us take $\varepsilon > 0$ and $[\alpha, \beta] \subset (A, B)$. If $[\alpha, \beta] \subset (A, x_0)$ or $[\alpha, \beta] \subset (x_0, B)$, then for these segments Property 2 will be satisfied by the conditions of the theorem. Suppose that $[\alpha, \beta] \subset (A, B)$ and $A < \alpha < x_0 < \beta < B$. By hypothesis $|f_n(t)| \geqslant M$. We must show the uniform convergence of the sequence $\{F_n(x)\}$ in the segment $[\alpha, \beta]$. Consequently, we must consider $x \in [\alpha, \beta]$. For such $x$ we have

$$|F_n(x)-F_m(x)| = \left|\int\limits_{x_0}^{x} \{f_n(t)-f_m(t)\}\,\mathrm{d}t\right|$$

$$\leqslant \left|\int\limits_{x_0}^{x} |f_n(t)-f_m(t)|\,\mathrm{d}t\right| \leqslant \int\limits_{\alpha}^{\beta} |f_n(t)-f_m(t)|\,\mathrm{d}t$$

$$= \int\limits_{\alpha}^{x_0-\frac{\varepsilon}{6M}} |f_n-f_m|\,\mathrm{d}t + \int\limits_{x_0-\frac{\varepsilon}{6M}}^{x_0+\frac{\varepsilon}{6M}} |f_n-f_m|\,\mathrm{d}t + \int\limits_{x_0+\frac{\varepsilon}{6M}}^{\beta} |f_n-f_m|\,\mathrm{d}t$$

Since the sequence $\{f_n(t)\}$ converges uniformly in the segments $[\alpha, x_0-\varepsilon/6M]$ and $[x_0+\varepsilon/6M, \beta]$, we can find a positive integer $n_0$ such that the following inequalities will be satisfied in the two segments:

$$|f_n(t)-f_m(t)| < \frac{\varepsilon}{3(\beta-\alpha)} \quad \text{for all } m, n \geqslant n_0$$

For $n$, $m \geqslant n_0$ we then have

$$|F_n(x)-F_m(x)| < \frac{\varepsilon}{3(\beta-\alpha)}\left(x_0-\frac{\varepsilon}{6M}-\alpha\right) + \frac{\varepsilon}{3}+\frac{\varepsilon}{3(\beta-\alpha)}\left(\beta-x_0-\frac{\varepsilon}{6M}\right) < \varepsilon$$

Consequently, the sequence $\{F_n(x)\}$ converges uniformly in any segment $[\alpha, \beta] \subset (A, B)$. Property 2 is therefore satisfied and this proves the theorem.

*Remark* If the sequence $\{f_n(x)\}$ is fundamental, the sequence $\left\{\int_{x_0}^{x} f_n(t)\,\mathrm{d}t\right\}$ is also fundamental.

*Example 1* Consider the sequence of functions $\{g_n(x)\}$

$$g_n(x) = \frac{1}{1+\mathrm{e}^{-nx}}$$

and the interval $(-\infty, \infty)$ (Fig. A.1).

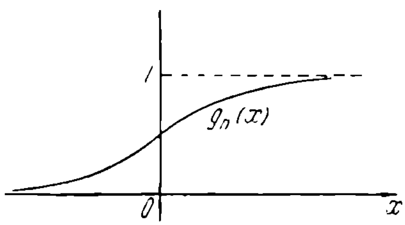



*Fig. A.1*

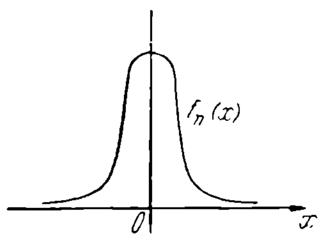



*Fig. A.2*

This sequence is uniformly bounded by the number 1. In any segment $[\alpha, \beta] \subset (-\infty, 0)$ it converges uniformly to zero. In any segment $[\alpha, \beta] \subset (0, \infty)$ it converges uniformly to unity. Therefore, Theorem 3 applies. Consequently, the sequence $\{g_n(x)\}$ is a fundamental sequence.

*Example 2* Consider the sequence of functions $\{f_n(x)\}$

$$f_n(x) = \sqrt{\frac{n}{\pi}}\,e^{-nx^2}$$

and the interval $(-\infty, \infty)$ (Fig. A.2). In any segment $[\alpha, \beta] \subset (-\infty, 0)$ or $[\alpha, \beta] \subset (0, \infty)$, the sequence converges uniformly to zero. However, it is not uniformly bounded. The sequence of functions $\{F_n(x)\}$, where $F_n(x) = \int\limits_{-\infty}^{x} f_n(t)\,dt$ will also converge uniformly in any segment $[\alpha, \beta]$ within the intervals $(-\infty, 0)$ or $(0, \infty)$ and is uniformly bounded (by the number 1) in the interval $(-\infty, \infty)$. Consequently, by Theorem 3 it is a fundamental sequence. If this is so, then by Theorem 2 the sequence $\{f_n(t)\}$ is also fundamental.

*Example 3* Consider the sequence of functions $\{\varphi_n(x)\}$, where $\varphi_n(x)$ is a piecewise-linear continuous function equal to zero outside the interval $(-1/n, 1/n)$ (Fig. A.3). In any segment $[\alpha, \beta]$ within the intervals $(-\infty, 0)$ and $(0, \infty)$ it converges uniformly to zero. However, it is not uniformly bounded. The sequence of functions

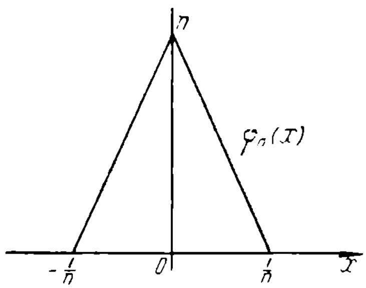



Fig. A.3

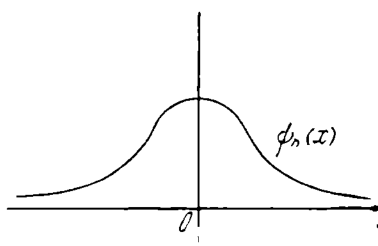



Fig. A.4

$\{\Phi_n(x)\}$, $\Phi_n(x) = \int\limits_{-\infty}^{x} \varphi_n(t)\,dt$, will also converge uniformly in any segment $[\alpha, \beta]$ in the intervals $(-\infty, 0)$ and $(0, \infty)$ and is uniformly bounded (by the number 1) in the interval $(-\infty, \infty)$. Consequently, by Theorem 3 it is a fundamental sequence. If this is so, it follows from Theorem 2 that the sequence $\{\varphi_n(x)\}$ is also a fundamental sequence.

*Example 4* Consider the sequence of functions $\{\varphi_n(x)\}$ where

$$\varphi_n(x) = \frac{\frac{\pi}{2}}{\left(\frac{1}{n}\right)^2 + x^2}$$

and the interval $(-\infty, \infty)$ (Fig. A.4). Since the sequence of functions $\{\Psi_n(x)\}$, where $\Psi_n(x) = \int_{-\infty}^{x} \psi_n(t)\mathrm{d}t$, satisfies the conditions of Theorem 3, this sequence is also fundamental. Consequently, by Theorem 2 the sequence $\{\psi_n(x)\}$ is a fundamental sequence.

**A.3** Two fundamental sequences $\{f_n(x)\}$ and $\{g_n(x)\}$ are called equivalent, $\{f_n(x)\} \sim \{g_n(x)\}$, if there exists an integer $k \geqslant 0$ and two other sequences $\{F_n(x)\}$, $\{G(x)\}$ such that

1. $F_n^{(k)} = f_n(x)$, $G_n^{(k)}(x) = g_n(x)$;
2. in any segment $[\alpha, \beta] \subset (A, B)$ the sequence $\{F_n(x) - G_n(x)\}$ converges uniformly to zero:

$$F_n(x) - G_n(x) \rightrightarrows 0$$

Thus, the sequences in Examples 2, 3 and 4 are equivalent to one another. They are also equivalent to the sequence $\{g_n'(x)\}$ (Example 1).

*Definition* Each class of equivalent fundamental sequences defines a generalised function $f(x)$ which can be represented by any other sequence of this class. We shall also say that a fundamental sequence $\{f_n(x)\}$ defines a generalised function $f(x)$ and write $f(x) \equiv \{f_n(x)\}$.

Thus, the sequence $\{g_n(x)\}$ (Example 1) defines the unit step function $\eta(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$ Consequently, the unit step function $\eta(x)$ is a generalised function.

In view of the lemma in Section A.4, and Theorem 1, any function which is continuous in $(A, B)$ is a generalised function.

The sequences $\{g_n'(x)\}$, $\{f_n(x)\}$, $\{\varphi_n(x)\}$, $\{\psi_n(x)\}$ in Examples 1–4 define a generalised function $\delta(x)$ (with a singularity at $x = 0$). This is called the Dirac $\delta$ function. It is clear that the $\delta$ function $\delta(x)$ is an even function, i.e. $\delta(-x) = \delta(x)$.

**A.4** A linear combination $\alpha f + \beta\varphi$ ($\alpha$ and $\beta$ are constants) of two generalised functions $f(x)$ and $\varphi(x)$, specified by the fundamental sequences $\{f_n(x)\}$ and $\{\varphi_n(x)\}$, is defined as the generalised function $F(x)$ corresponding to the fundamental sequence $\{\alpha f_n(x) + \beta\varphi_n(x)\}$. In particular, the sum and difference of two generalised functions $f(x)$ and $\varphi(x)$ is also a generalised function.

The product of a generalised function $\eta(x-x_0)$, i.e. the unit function, which is continuous in the segment $[a, b]$, and the function $\varphi(x)$, which will be written $\eta(x-x_0)\varphi(x)$, will be defined as the

generalised function defined by the fundamental sequence $\{g_n(x-x_0)\tilde{\varphi}(x)\}$, where $g_n(x)$ is the function in Example 1, and

$$\tilde{\varphi}(x) = \begin{cases} \varphi(b), & x \geqslant b \\ \varphi(x), & a \leqslant x \leqslant b \\ \varphi(a), & x \leqslant a \end{cases}$$

It is clear that

$$\eta(x-x_0)\varphi(x) = \begin{cases} \varphi(x), & x > x_0 \\ 0, & x < x_0 \end{cases}$$

A function $\varphi(x)$ which is piecewise-continuous in $[a, b]$ and has singularities at $x_1, x_2, \ldots, x_k (a < x_1 < x_2 < \ldots < x_k < b)$ can be written in the form

$$\begin{aligned} \varphi(x) = {} & [\eta(x-a)-\eta(x-x_1)]\,\tilde{\varphi}_0(x)+ \ldots \\ & + [\eta(x-x_i)-\eta(x-x_{i+1})]\,\tilde{\varphi}_i(x)+ \ldots \\ & + [\eta(x-x_k)-\eta(x-b)]\,\tilde{\varphi}_k(x) \end{aligned}$$

where

$$\tilde{\varphi}_i(x) = \left\{ \begin{array}{ll} \varphi(x_{i+1}), & x \geqslant x_{i+1} \\ \varphi(x), & x_i \leqslant x \leqslant x_{i+1} \\ \varphi(x_i), & x \leqslant x_i \end{array} \right\} \begin{array}{l} (i = 0, 1, 2, \ldots, k) \\ x_0 = a, x_{k+1} = b \end{array}$$

$\tilde{\varphi}_i(x)$ are continuous in $(-\infty, \infty)$. Their products by unit step functions $\eta(x-x_i)$ are generalised functions. Consequently, an arbitrary piecewise-continuous function is a linear combination of generalised functions and is, therefore, also a generalised function.

**A.5** Let $\{\varepsilon_n\}$ be a sequence of positive numbers tending to zero.

*Definition* A sequence of continuous functions $\{\delta_n(x)\}$ is called a $\delta$ sequence if these functions have the following properties:

1. $\delta_n(x) > 0$ in $(-\varepsilon_n, \varepsilon_n)$ and zero elsewhere;
2. $\delta_n(x)$ has derivatives of all orders everywhere;
3. $\int_{-\infty}^{\infty} \delta_n(t)\,\mathrm{d}t = 1$.

Consider the function

$$\alpha(x) = \begin{cases} \mathrm{e}^{-1/x}, & x > 0 \\ 0, & x \leqslant 0 \end{cases}$$

It has derivatives of all orders. The function

$$\beta_n(x) = \alpha(x+\varepsilon_n)\,\alpha(\varepsilon_n-x)$$

is positive in $(-\varepsilon_n, \varepsilon_n)$ and zero elsewhere and possesses derivatives of all orders. Let

$$\|\beta_n\| = \int_{-\infty}^{\infty} \beta_n(t)\,\mathrm{d}t$$

We then have

$$\int_{-\infty}^{\infty} \frac{1}{\|\beta_n\|}\beta_n(t)\,\mathrm{d}t = 1$$

Consequently, the sequence $\left\{\dfrac{\beta_n(x)}{\|\beta_n\|}\right\}$ is a $\delta$ sequence.

*Theorem 4* Any $\delta$ sequence is a fundamental sequence.

*Proof* The sequence of functions

$$\gamma_n(x) = \int_{-\infty}^{x} \delta_n(t)\,\mathrm{d}t$$

is uniformly bounded, since $0 \leqslant \gamma_n(x) \leqslant 1$, and in any segment $[\alpha, \beta]$ which does not contain $x = 0$ converges uniformly to zero if $[\alpha, \beta] \subset (-\infty, 0)$, and to unity if $[\alpha, \beta] \subset (0, \infty)$. In point of fact, in the former case $\{[\alpha, \beta] \subset (-\infty, 0)\}$ we can find a number $n_0$ such that for all $n > n_0$ we have $\varepsilon_n < |\beta|$. Consequently, the segment $[\alpha, \beta]$ will lie to the left of all intervals $(-\varepsilon_n, \varepsilon_n)$. Therefore, all functions $\delta_n(x)$ with $n > n_0$ will be zero in $[\alpha, \beta]$. Hence, it follows that $\gamma_n(x) \equiv 0$ in $[\alpha, \beta]$ for $n > n_0$. In the second case $\{[\alpha, \beta] \subset (0, \infty)\}$ we can find a number $n_0$ such that for $n > n_0$ we have $\varepsilon_n < \alpha$. Consequently, the segment $[\alpha, \beta]$ will lie to the right of all intervals $(-\varepsilon_n, \varepsilon_n)$, $n > n_0$. Therefore, for $x \in [\alpha, \beta]$ and $n > n_0$,

$$\gamma_n(x) = \int_{-\infty}^{x} \delta_n(t)\,\mathrm{d}t = \int_{-\varepsilon_n}^{\varepsilon_n} \delta_n(t)\,\mathrm{d}t = 1$$

Hence Theorem 3 applies and the sequence $\{\gamma_n(x)\}$ is a fundamental sequence. If this is so, it follows that by Theorem 2 the sequence $\{\delta_n(x)\}$ is also a fundamental sequence, and this proves the theorem.

*Remark* It is clear that for all constants $C$ the sequence $\{C\delta_n(x)\}$ is a fundamental sequence. It is readily shown that all $\delta$ sequences are equivalent to each other and to the sequences in Examples 2–4.

The reader is recommended to prove this. Consequently, any $\delta$ sequence $\{\delta_n(x)\}$ defines the $\delta$ function $\delta(x)$. The $\delta$ sequence $\{\delta_n(x-x_0)\}$ defines the $\delta$ function $\delta(x-x_0)$.

Multi-dimensional $\delta$ sequences are defined in a similar way. For example, in three-dimensional space the $\delta$ sequence is defined as being of the form

$$\{\delta_n(x)\,\delta_n(y)\,\delta_n(z)\}$$

i.e. as consisting of the products $\delta_n(x)\,\delta_n(y)\,\delta_n(z)$, whereas the $\delta$ function in three-dimensional space is defined by $\delta$ sequences of this form. Therefore, by definition, $\delta(M) = \delta(x)\,\delta(y)\,\delta(z)$ and

$$\begin{aligned}\delta(M, M_0) &\equiv \{\delta_n(x-x_0)\,\delta_n(y-y_0)\,\delta_n(z-z_0)\}\\ &= \delta(x-x_0)\,\delta(y-y_0)\,\delta(z-z_0)\end{aligned}$$

where $x, y, z$ are the coordinates of the point $M$ and $x_0, y_0, z_0$ are the coordinates of the point $M_0$.

We have shown above that the sequences $\{A\,\delta_n(x-x_0)\}$ are fundamental sequences. It is possible to prove a more general statement, namely that for all functions $\varphi(x)$ which are continuous in the neighbourhood $O(x_0)$ of the point $x = x_0$, the sequence $\{\varphi(x)\,\delta_n(x-x_0)\}$ is fundamental in $O(x_0)$ and is equivalent to the sequence $\{\varphi(x_0)\,\delta_n(x-x_0)\}$.

*Proof* Consider the functions

$$F(x) = \int_{-\infty}^{x} \varphi(t)\,\delta_n(t-x_0)\,\mathrm{d}t \quad \text{and} \quad \Phi_n(x) = \int_{-\infty}^{x} \varphi(x_0)\,\delta_n(t-x_0)\,\mathrm{d}t$$

It is clear that $F_n'(x) = \varphi(x)\,\delta_n(x-x_0)$, $\Phi_n'(x) = \varphi(x_0)\,\delta_n(x-x_0)$ for all points of $O(x_0)$. For any $\varepsilon > 0$ we can find a number $n_0(\varepsilon)$ such that

$$|\varphi(x)-\varphi(x_0)| < \varepsilon \quad \text{for} \quad x \in (x_0-\varepsilon_n, x_0+\varepsilon_n) \quad \text{and} \quad n > n_0$$

Then for any $x \in O(x_0)$ and $n > n_0$

$$\begin{aligned}|F_n(x)-\Phi_n(x)| &= \left|\int_{-\infty}^{x} \{\varphi(t)-\varphi(x_0)\}\,\delta_n(t-x_0)\,\mathrm{d}t\right|\\ &\leqslant \int_{-\infty}^{x} |\varphi(t)-\varphi(x_0)|\,\delta_n(t-x_0)\,\mathrm{d}t\\ &\leqslant \int_{x_0-\varepsilon_n}^{x_0+\varepsilon_n} |\varphi(t)-\varphi(x_0)|\,\delta_n(t-x_0)\,\mathrm{d}t\\ &< \varepsilon \int_{x_0-\varepsilon_n}^{x_0+\varepsilon_n} \delta_n(t-x_0)\,\mathrm{d}t = \varepsilon\end{aligned}$$

i.e.

$$|F_n(x)-\Phi_n(x)| < \varepsilon \tag{A.1}$$

Since the sequence $\{\Phi_n(x)\}$ converges uniformly $[(\Phi_n(x) = \varphi(x_0)\gamma_n(x-x_0)$; see proof of Theorem 4], it follows that the sequence $\{F_n(x)\}$ converges uniformly in any segment $[\alpha, \beta] \subset O(x_0)$. From this and from the inequality given by (A.1) it follows that the sequence $\{\varphi(x)\ \delta_n(x-x_0)\}$ is fundamental in $O(x_0)$ and is equivalent to the sequence $\{\varphi(x_0)\ \delta_n(x^2-x_0)\}$.

The generalised function $\varphi(x)\ \delta(x-x_0)$ specified by the fundamental sequence $\{\varphi(x)\ \delta_n(x-x_0)\}$ is defined as the product of the function $\varphi(x)$ which is continuous in the neighbourhood of the point $x = x_0$ and the $\delta$ function $\delta(x-x_0)$.

The above statement means that

$$\varphi(x)\,\delta(x-x_0) = \varphi(x_0)\,\delta(x-x_0)$$

**A.6** *Theorem 5* Equivalent fundamental sequences defining the generalised function $f(x)$ include fundamental sequences of differentiable functions (polynomials).

Let us first establish the following lemma.

*Lemma* For any function $F(x)$ which is continuous in $(A, B)$ there exists a sequence of polynomials $\{P_n(x)\}$ which converges uniformly to $F(x)$ in any segment $[\alpha, \beta] \subset (A, B)$.

*Proof* Let $\{A_n\}$ be a decreasing sequence of numbers converging to $A$, and let $\{B_n\}$ be an increasing number of zeros converging to $B$. By the Weierstrass theorem there exists a polynomial $P_n(x)$ such that $|F(x)-P_n(x)| < 1/n$ for all points $x$ in $[A_n, B_n]$. The sequence of such polynomials has the property indicated in the lemma. In fact, for any positive number $\varepsilon$ and any segment $[\alpha, \beta] \subset (A, B)$ one can find a positive integer $n_0$ such that for all $n > n_0$

$$\frac{1}{n} < \varepsilon \quad \text{and} \quad [\alpha, \beta] \subset [A_n, B_n]$$

It then follows that for all $n > n_0$ and all $x \in |\alpha, \beta|$,

$$|P_n(x)-F(x)| < \frac{1}{n} < \varepsilon$$

which indicates a uniform convergence of the sequence $\{P_n(x)\}$ to the function $F(x)$ in the segment $[\alpha, \beta]$. This proves the lemma.

*Proof of the theorem* Let $\{f_n(x)\}$ be a fundamental sequence defining the generalised function $f(x)$. By definition of a fundamental sequence there exists an integer $k \geqslant 0$ and a sequence of continuous functions $\{F_n(x)\}$ which converges in $(A, B)$ to a function $F(x)$, which are such that

$$F_n^{(k)}(x) = f_n(x)$$

The function $F(x)$ is continuous in $(A, B)$ since the sequence $\{F_n(x)\}$ converges uniformly to $F(x)$ in any segment $[\alpha, \beta] \subset (A, B)$. In accordance with the above lemma, there exists a sequence of polynomials $\{P_n(x)\}$ which converges uniformly to $F(x)$ in any segment $[\alpha, \beta] \subset (A, B)$. Therefore, by definition of equivalent fundamental sequences, the sequence of polynomials

$$\{p_n(x)\} \quad \text{where} \quad p_n(x) = P_n^{(k)}(x)$$

is a fundamental sequence which is equivalent to $\{f_n(x)\}$. This proves the theorem.

Therefore, it can always be considered that the generalised function $f(x)$ is defined by the fundamental sequence of differentiable functions $\{f_n(x)\}$.

*Definition* The $m$-th order derivative of a generalised function $f(x) \equiv \{f_n(x)\}$ is defined as the generalised function

$$f^{(m)}(x) \equiv \{f_n^{(m)}(x)\}$$

which is defined by the fundamental sequence of $m$-th order derivatives $\{f_n^{(m)}(x)\}$. Thus, in particular, the $\delta$ function $\delta(x)$ possesses derivatives of all orders. For example, $\delta'(x)$ is defined by the fundamental sequence $\{\delta_n'(x)\}$. The derivative of the unit step function $\eta(x)$ is the generalised function

$$\eta'(x) = \delta(x)$$

since the sequence $\{g_n'\}$ of Example 1 is equivalent to $\{f_n(x)\}$ of Example 2, which defines the $\delta$ function. Consequently, we may write

$$\eta(x) = \int_{-\infty}^{x} \delta(t)\, \mathrm{d}t$$

**A.7** The integral of the product of $\delta(x-x_0)$ and of an arbitrary continuous function $\varphi(x)$, i.e.

$$\int_a^b \varphi(x)\, \delta(x-x_0)\, \mathrm{d}x$$

is defined as the limit

$$\lim_{n\to\infty}\int_a^b \varphi(x)\,\delta_n(x-x_0)\,dx$$

where $\{\delta_n(x-x_0)\}$ is any $\delta$ sequence defining the $\delta$ function $\delta(x-x_0)$. We shall show that this limit exists and that the following formula is valid

$$\int_a^b \varphi(x)\,\delta(x-x_0)\,dx = \begin{cases}\varphi(x_0), & \text{if} \quad x_0 \in (a, b)\\ 0, & \text{if} \quad x_0 \notin [a, b]\end{cases}$$

It may be assumed that the functions $\delta_n(x-x_0)$ which form the $\delta$ sequence are even with respect to $x = x_0$. For $x_0 = a$ or $x_0 = b$, we then have

$$\int_a^b \varphi(x)\,\delta(x-x_0)\,dx = \frac{1}{2}\varphi(x_0) \quad \text{(Prove this!)}$$

Suppose that $x_0 \notin [a, b]$. We can then find a number $n_0$ such that for all $n > n_0$ we have $\varepsilon_n < \min\{|x_0-a|, |x_0-b|\}$. Consequently, for $n > n_0$ we have $\delta_n(x-x_0) \equiv 0$ in $[a, b]$.

Therefore $\int_a^b \varphi(x)\,\delta_n(x-x_0)\,dx = 0$. Consequently

$$\int_a^b \varphi(x)\delta(x-x_0)\,dx = 0$$

Let $x_0 \in (a, b)$. We can then find a number $n_0$ such that for $n > n_0$ the intervals $(x_0-\varepsilon_n, x_0+\varepsilon_n)$ will lie entirely in the segment $[a, b]$. Consequently, for such $n$,

$$\int_a^b \varphi(x)\,\delta_n(x-x_0)\,dx = \int_{x_0-\varepsilon_n}^{x_0+\varepsilon_n} \varphi(x)\,\delta_n(x-x_0)\,dx$$

Let us apply the mean value theorem to the last integral. We obtain

$$\int_{x_0-\varepsilon_n}^{x_0+\varepsilon_n} \varphi(x)\,\delta_n(x-x_0)\,dx = \varphi(\xi_n)\int_{x_0-\varepsilon_n}^{x_0+\varepsilon_n} \delta_n(x-x_0)\,dx = \varphi(\xi_n)$$

where $\xi_n \in [x_0-\varepsilon_n, x_0+\varepsilon_n]$. As $n \to \infty$, we have $\xi_n \to x_0$. Therefore, since the function $\varphi(x)$ continues in $[a, b]$, we obtain

$$\lim_{n\to\infty}\int_a^b \varphi(x)\,\delta_n(x-x_0)\,dx = \lim_{n\to\infty}\varphi(\xi_n) = \varphi(x_0)$$

This proves the above statement. In particular, for $\varphi(x) \equiv 1$, we have

$$\int_a^b \delta(x-x_0)\,\mathrm{d}x = \begin{cases} 1, & x_0 \in [a, b] \\ 0, & x_0 \notin [a, b] \end{cases}$$

The formula

$$\int_D \varphi(M)\,\delta(M, M_0)\,\mathrm{d}\tau_M = \begin{cases} \varphi(M_0), & \text{if} \quad M_0 \in D, \\ 0, & \text{if} \quad M_0 \notin \bar{D} \end{cases} \tag{A.2}$$

can be proved in a similar way for any function $\varphi(M)$ which is continuous in $D$.

*Remark* Each of the sequences $\{f_n\}$, $\{\varphi_n\}$, $\{\psi_n\}$ of Examples 2–4, which define the $\delta$ function $\delta(x)$, converges to zero at any point $x \neq 0$ and to infinity at $x = 0$. In view of this, we may write

$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases}$$

The function $\delta(x)$ becomes infinite at $x = 0$ so that $\int_{-a}^{a} \delta(x)\,\mathrm{d}x = 1$ for any $a > 0$. The expression (A.2) is often used to define the $\delta$ function as a functional.

**A.8** A function $\omega(x)$ will be defined as localised if it is identically equal to zero outside a given interval $(a, b)$. A localised function will be called smooth if it is continuous and possesses continuous derivatives of all orders everywhere.

Let $f(x)$ be a continuous or locally integrable, i.e. integrable in any finite interval, function. The convolution of $f(x)$ and $\varphi(x)$ is defined by

$$f(x) * \varphi(x) = \int_{-\infty}^{\infty} f(x-t)\,\varphi(t)\,\mathrm{d}t \tag{A.3}$$

It is clear that

$$f(x) * \varphi(x) = \varphi(x) * f(x) = \int_{-\infty}^{\infty} f(t)\,\varphi(x-t)\,\mathrm{d}t \tag{A.4}$$

Since $\varphi(x)$ is a localised function, the convolution can also be written in the form

$$f * \varphi = \int_a^b f(x-t)\,\varphi(t)\,\mathrm{d}t \tag{A.5}$$

within the interval outside which $\varphi(x) \equiv 0$. The following are the simplest properties of the convolution.

*Property 1* If the function $\varphi(x)$ possesses everywhere continuous derivatives up to $k$-th order, then the convolution $f * \varphi$ has derivatives everywhere up to $k$-th order and

$$\frac{d^p}{dx^p}[f(x) * \varphi(x)] = [f * \varphi]^{(p)} = f(x) * \varphi^{(p)}(x) \qquad (p = 1, 2, \ldots, k) \tag{A.6}$$

This follows directly from (A.4) and from the fact that the function $\varphi(x)$ is localised. If the function $f(x)$ has continuous derivatives up to order $k$ and $\varphi(x)$ is integrable and finite, then

$$(f * \varphi)^{(p)} = f^{(p)}(x) * \varphi(x) \qquad (p = 1, 2, \ldots, k) \tag{A.7}$$

*Property 2* If the sequence of continuous functions $\{f_n(x)\}$ converges uniformly in the range $a_0 - b \leqslant x \leqslant b_0 - a$ to the function $f(x)$, then for any continuous localised function $\varphi(x)$ which is identically zero outside the range $(a, b)$, the sequence $\{f_n(x) * \varphi(x)\}$ converges uniformly to the function $f(x) * \varphi(x)$ in the range $[a_0, b_0]$.

The validity of this follows directly from (A.5).

Let $(A', B')$ represent the interval consisting of points $x'$ such that $[x'-b, x'-a] \subset (A, B)$.

*Property 3* If the sequence $\{f_n(x)\}$ is fundamental in $(A, B)$ and $\varphi(x)$ is localised and continuous [$\varphi(x) \equiv 0$ outside $(a, b)$], then the sequence $\{f_n(x) * \varphi(x)\}$ is fundamental in $(A', B')$.

*Proof* Let $[\alpha, \beta] \subset (A' B')$. The segment $[\alpha - b, \beta - a]$ then belongs to $(A, B)$. Since $\{f_n(x)\}$ is a fundamental sequence, there exists an integer $k \geqslant 0$ and another sequence $\{F_n(x)\}$ such that $F_n^{(k)} = f_n(x)$ and $\{F_n(x)\}$ converges uniformly in $[\alpha - b, \beta - a]$. We have

$$f_n(x) * \varphi(x) = F_n^{(k)}(x) * \varphi(x)$$

and, using Equation (A.7), we obtain

$$f_n(x) * \varphi(x) = [F_n(x) * \varphi(x)]^{(k)} \tag{A.8}$$

In view of Property 2, the sequence $\{F_n(x) * \varphi(x)\}$ converges uniformly in $[\alpha, \beta]$. Hence, and from Equation (A.8), it follows that the sequence $\{f_n(x) * \varphi(x)\}$ is a fundamental sequence.

*Remark* If $\varphi(x)$ is a smooth function, then Equation (A.8) can be written in the form

$$f_n(x) * \varphi(x) = F_n(x) * \varphi^{(k)}(x) \tag{A.9}$$

Since the sequence $\{F_n(x)\}$ converges uniformly in any segment $[\alpha, \beta] \subset (A, B)$, then by Property 2, the sequence $\{F_n(x) * \varphi^{(k)}(x)\}$ converges uniformly in any segment $[\alpha+b, \beta+a]$ belonging to $(A', B')$. Therefore, we have the following property.

*Property 4* If the sequence $\{f_n(x)\}$ is fundamental in $(A, B)$, and $\varphi(x)$ is a smooth finite function [$\varphi(x) \equiv 0$ outside $(a, b)$], then the sequence $\{f_n(x) * \varphi(x)\}$ converges uniformly in any segment $[\alpha', \beta'] \subset (A', B')$.

This leads naturally to the following.

*Definition* The convolution of an arbitrary generalised function $f(x)$, defined by the fundamental sequence $\{f_n(x)\}$, and a localised continuous function $\varphi(x)$, is defined as the generalised function $f(x) * \varphi(x)$ defined by the fundamental sequence $\{f_n(x) * \varphi(x)\}$. It is clear that

$$f(x) * \varphi(x) = \varphi(x) * f(x)$$

The convolution of the generalised function $f(x)$ and an arbitrary localised integrable function $\varphi(x)$ is defined in a similar way. Moreover, $(f * \varphi)^{(p)} = f^{(p)} * \varphi$, where $f^{(p)}$ is the $p$-th order derivative of the generalised function. In particular, if $\varphi(x)$ is an arbitrary localised and everywhere continuous function,

$$\delta(x) * \varphi(x) = \varphi(x) * \delta(x)$$

*Property 5* For a localised and everywhere continuous function $\varphi(x)$,

$$\varphi(x) * \delta(x) \equiv \varphi(x)$$

*Lemma* Let $\{\delta_n(x)\}$ be a $\delta$ sequence and $\varphi(x)$ be a function which is continuous in $(A, B)$. The sequence $\{\varphi(x) * \delta_n(x)\}$ will then converge to $\varphi(x)$ uniformly in any segment $[\alpha, \beta] \subset (A, B)$.

*Proof* Let $[\alpha, \beta] \subset (A, B)$. For any $\varepsilon > 0$ one can then find $n_0(\varepsilon)$ such that for $n > n_0$ and all $x$ in $[\alpha, \beta]$ and all $t$ in $(-\varepsilon_n, \varepsilon_n)$,

$$|\varphi(x-t) - \varphi(x)| < \varepsilon$$

Consider the difference $\varphi(x) * \delta_n(x) - \varphi(x)$:

$$
\begin{aligned}
|\varphi(x) * \delta_n(x) - \varphi(x)| &= \left| \int_{-\infty}^{\infty} \varphi(x-t)\,\delta_n(t)\,\mathrm{d}t - \varphi(x) \right| \\
&= \left| \int_{-\infty}^{\infty} \varphi(x-t)\,\delta_n(t)\,\mathrm{d}t - \int_{-\infty}^{\infty} \varphi(x)\,\delta_n(t)\,\mathrm{d}t \right| \\
&\leqslant \int_{-\infty}^{\infty} |\varphi(x-t) - \varphi(x)|\,\delta_n(t)\,\mathrm{d}t \\
&= \int_{-\varepsilon_n}^{\varepsilon_n} |\varphi(x-t) - \varphi(x)|\,\delta_n(t)\,\mathrm{d}t
\end{aligned}
$$

For $n > n_0(\varepsilon)$ and any $x \in |\alpha, \beta|$, the last integral is less than

$$\varepsilon \int_{-\varepsilon_n}^{\varepsilon_n} \delta_n(t)\,\mathrm{d}t = \varepsilon$$

Therefore, for $n > n_0(\varepsilon)$ and $x \in [\alpha, \beta]$

$$|\varphi(x) * \delta_n(x) - \varphi(x)| < \varepsilon$$

This proves the lemma.

*Proof of Property 5* Since the functions $\varphi(x) * \delta_n(x)$ are continuous in $(A, B)$, and in accordance with the above lemma, the sequence $\{\varphi(x) * \delta_n(x)\}$, converges uniformly to the function $\varphi(x)$ in any segment $[\alpha, \beta] \subset (A, B)$, then by Theorem 1 it is fundamental and defines the generalised function $\varphi(x)$. On the other hand, by definition, the fundamental sequence $\{\varphi(x) * \delta_n(x)\}$ defines the convolution $\varphi(x) * \delta(x)$. Therefore,

$$\varphi(x) * \delta(x) = \varphi(x)$$

Moreover,

$$
\begin{aligned}
\varphi(x) = \lim_{n\to\infty} [\varphi(x) * \delta_n(x)] &= \lim_{n\to\infty} \int_{-\infty}^{\infty} \varphi(x-t)\,\delta_n(t)\,\mathrm{d}t \\
= \int_{-\infty}^{\infty} \varphi(x-t)\,\delta(t)\,\mathrm{d}t &= \int_{-\infty}^{\infty} \varphi(t)\,\delta(x-t)\,\mathrm{d}t
\end{aligned}
$$

where we have used the definition of an integral of the product of a continuous function and a $\delta$ function.

Therefore, the convolution of a $\delta$ function and an arbitrary continuous function $\varphi(x)$ can be written in the form

$$\varphi(x) * \delta(x) = \int_{-\infty}^{\infty} \varphi(x-t)\,\delta(t)\,\mathrm{d}t = \int_{-\infty}^{\infty} \varphi(t)\,\delta(x-t)\,\mathrm{d}t$$

*Property 6* The convolution of an arbitrary generalised function $f(x)$ and a smooth function $\varphi(x)$ has continuous derivatives of all orders.

*Proof* Let the generalised function $f(x)$ be defined by the fundamental sequence $\{f_n(x)\}$. By Property 4 the sequence $\{f_n(x) * \varphi(x)\}$ converges uniformly to the continuous function in any segment $[\alpha', \beta'] \subset (A', B')$. From Equation (A.9) and Property 4 it follows that the sequence $\{f_n * \varphi)^{(i)}\}$ of the $i$-th-order derivatives ($i = 1, 2, \ldots$) will also converge uniformly to continuous functions. Using the theorem on the term-by-term differentiation of sequences, we obtain Property 6.

It is possible to define the convolution of an arbitrary generalised function $f(x)$ and a $\delta$ function as the generalised function $f(x) * \delta(x)$, defined by the fundamental sequence

$$\{f(x) * \delta_n(x)\}$$

where $\{\delta_n(x)\}$ is an arbitrary $\delta$ sequence. We then have

$$f(x) * \delta(x) = f(x)$$

The following is a list of the most useful formulae and relationships involving the $\delta$ function. The reader is recommended either to prove these results for himself or to refer to specialist literature.

1. $\delta(-x) = \delta(x)$

2. $\delta(\alpha x) = \dfrac{1}{\alpha}\,\delta(x)$

3. $\delta(\varphi(x)) = \sum\limits_n \dfrac{\delta(x-x_n)}{|\varphi'(x_n)|}$ if $\varphi(x)$ has only simple zeros $x_n$.

4. $x\delta(x) = 0, \ \varphi(x)\delta(x) = \varphi(0)\delta(x), \ \varphi(a \pm x)\delta(x) = \varphi(a)\delta(x)$

5. $\displaystyle\int_{-\infty}^{\infty} \varphi(x)\delta(x-x_0)\,\mathrm{d}x = \varphi(x_0)$

6. $\displaystyle\int_{-\infty}^{\infty} \delta(x-t)\delta(s-t)\,\mathrm{d}t = \delta(x-s)$

7. $\delta'(x) = \dfrac{-1}{x}\delta(x)$

8. $\displaystyle\int_{-\infty}^{\infty} \varphi(x)\,\delta'(x-x_0)\,\mathrm{d}x = -\varphi'(x_0),$

if $\varphi(x)$ is continuous at $x = x_0$.

9. $\displaystyle\int_{-\infty}^{\infty} \delta'(x-t)\delta(s-t)\,\mathrm{d}t = \delta'(x-s)$

10. $\dfrac{\mathrm{d}^n}{\mathrm{d}x^n}\delta(x) = (-1)^n\dfrac{n!}{x^n}\delta(x)$

11. $\displaystyle\int_{-\infty}^{\infty} \varphi(x)\frac{\partial^n}{\partial x^n}\delta(x-s)\,\mathrm{d}x = (-1)^n\varphi^{(n)}(s),$

if $\varphi^{(n)}(x)$ is continuous at $x = s$.

12. $\displaystyle\iiint_{-\infty}^{\infty}\varphi(M)\delta(M, M_0)\,\mathrm{d}\tau_M = \varphi(M_0)$

13. The Fourier transform of the $\delta$ function $\delta(x-x_0)$ is given by

$$\tilde{\delta}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \delta(x-x_0)\,\mathrm{e}^{i\xi x}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{i\xi x_0}$$

and, consequently,

$$\delta(x-x_0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{\delta}(\xi)\,\mathrm{e}^{-i\xi x}\,\mathrm{d}\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-i\xi(x-x_0)}\,\mathrm{d}\xi$$

For the three-dimensional case

$$\delta(M, M_0) = \left(\frac{1}{2\pi}\right)^3\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-i[\xi(x-x_0)+\eta(y-y_0)+\zeta(z-z_0)]}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-i\xi(x-x_0)}\,\mathrm{d}\xi\,\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-i\eta(y-y_0)}\,\mathrm{d}\eta\,\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-i\zeta(z-z_0)}\,\mathrm{d}\zeta$$

$$= \delta(x-x_0)\,\delta(y-y_0)\,\delta(z-z_0).$$

Therefore

$$\delta(M, M_0) = \delta(x-x_0)\,\delta(y-y_0)\,\delta(z-z_0)$$

in Cartesian coordinates.

14. $\delta(M, M_0) = \frac{1}{r}\delta(r-r_0)\delta(\varphi-\varphi_0)$ in polar coordinates on a plane.

15. $\delta(M, M_0) = \frac{1}{r^2}\delta(r-r_0)\frac{1}{\sin\theta}\delta(\theta-\theta_0)\delta(\varphi-\varphi_0)$ in spherical coordinates.

16. $\delta(M, M_0) = \frac{\delta(q_1-q_1^0)\,\delta(q_2-q_2^0)\,\delta(q_3-q_3^0)}{h_1 h_2 h_3}$ in arbitrary orthogonal curvilinear coordinates $(q_1, q_2, q_3)$, where $h_1$, $h_2$, and $h_3$ and the Lamé coefficients.

The generalised functions $\delta_+(x)$ and $\delta_-(x)$ are also encountered and are defined formally by

$$\delta_+(x) = \frac{1}{2\pi}\int_0^\infty e^{-i\xi x}\,d\xi, \quad \delta_-(x) = \frac{1}{2\pi}\int_{-\infty}^0 e^{-i\xi x}\,d\xi = \frac{1}{2\pi}\int_0^\infty e^{i\xi x}\,d\xi$$

It is clear that $\delta_+(x)+\delta_-(x) = \delta(x)$ and $\delta_+(-x) = \delta_-(x)$.

# Answers and Solutions

## CHAPTER 1

**1.** **(a)** $u_{\xi\eta}-0.5(1/\xi)u_\xi = 0 \quad (\xi = xy, \quad \eta = y/x)$;

**(b)** $\Delta u-u_\xi-u_\eta = 0 \quad (\xi = y^2, \quad \eta = x^2)$;

**(c)** $u_{\eta\eta} = 0 \quad (\xi = y/x, \quad \eta = y)$;

**(d)** $u_{\xi\eta}+\dfrac{0.5}{\xi-\eta}(u_\xi-u_\eta) = 0$ for $y < 0 \quad (\xi = x+2\sqrt{-y},$
$\eta = x-2\sqrt{-y})$, $\Delta u = 0$ for $y > 0 \quad (\xi = x, \quad \eta = 2\sqrt{y})$.

**2.** **(a)** $v_{\eta\eta}-2v_\xi = 0, \quad u = v\,e^{\mu\xi+\nu\eta}, \quad \mu = 1.1875, \quad \nu = 0.25$
$(\xi = y-x, \quad \eta = y+x)$;

**(b)** $v_{\xi\eta}+\dfrac{1}{32}v = 0, \quad u = v\,e^{\mu\xi+\nu\eta}, \quad \mu = -0.25, \quad \nu = -7/8$
$(\xi = y-3x, \quad \eta = y-x)$;

**(c)** $v_{\xi\xi}+v_{\eta\eta}-1.5v = 0, \quad u = v\,e^{-\xi-\eta} \quad (\xi = 2y-x, \quad \eta = x)$.

## CHAPTER 2

**1.** $a^2u_{xx}+g = u_{tt}, \quad u(x, 0) = 0, \quad u_t(x, 0) = v_0, \quad u(0, t) = u_x(l, t) = 0$.

**2.** $a^2u_{xx}-\beta u_t = u_{tt}, \qquad u(x, 0) = \varphi(x), \qquad u_t(x, 0) = \varphi_1(x)$,
$u(0, t) = u(l, t) = 0$.

**3.** $\partial/\partial x\,(ESu_{xx}) = \rho S u_{tt}, \qquad u(x, 0) = \varphi(x), \qquad u_t(x, 0) = \varphi_1(x)$,
$\alpha_1 u_x(0, t)-\beta_1 u(0, t) = 0, \qquad \alpha_2 u_x(l, t)+\beta_2 u(l, t) = 0$.

4. $\partial/\partial x\,[(l-x)u_x] = (1/g)u_{tt}$, $u(x,0) = \varphi(x)$, $u_t(x,0) = \varphi_1(x)$, $u(0,t) = 0$, $|u(l,t)| \leqslant M$.

5. $g\partial/\partial x\,[(l-x)u_x]+\omega^2 u = u_{tt}$, additional conditions of Problem 4.

6. $\omega^2\partial/\partial x\,[(l^2-x^2)u_x] = \rho u_{tt}$, additional conditions of Problem 4.

7. $a^2\theta_{xx} = \theta_{tt}$, $\theta(x,0) = \varphi(x)$, $\theta_t(x,0) = \varphi_1(x)$, $\theta(0,t) = \theta(l,t) = 0$.

8. $a^2u_{xx}+(H/c\rho)I(t) = u_{tt}$, $u(x,0) = u_t(x,0) = 0$, $u(0,t) = u(l,t) = 0$, $c$ is the velocity of light.

9. $u(x,t) = \begin{cases} u_1(x,t), & x<0 \\ u_2(x,t), & x>0 \end{cases}$ $\quad a_i^2(u_i)_{xx} = (u_i)_{tt} \quad (i = 1, 2)$,
$u(x,0) = \varphi(x)$, $u_t(x,0) = \varphi_1(x)$, $a_i^2 = E_i/\rho_i \quad (i = 1, 2)$,
$u_1(0,t) = u_2(0,t)$, $E_1u_{1x}(0,t) = E_2u_{2x}(0,t)$.

10. $Tu_{xx} = [\rho+m_0\delta(x-x_0)]u_{tt}$, $u(x,0) = \varphi(x)$, $u_t(x,0) = \varphi_1(x)$, $u(0,t) = u(l,t) = 0$.
Or:
$u(x,t) = \begin{cases} u_1(x,t), & x<x_0 \\ u_2(x,t), & x>x_0 \end{cases}$ $\quad a^2(u_i)_{xx} = (u_i)_{tt} \quad (i = 1, 2)$,
$u(x,0) = \varphi(x)$, $u_t(x,0) = \varphi_1(x)$, $u_1(0,t) = 0$, $u_2(l,t) = 0$,
$u_1(x_0,t) = u_2(x_0,t)$, $T[u_{2x}(x_0,t)-u_{1x}(x_0,t)] = m_0u_{tt}(x_0,t)$.

11. $Tu_{xx}+F(t)\delta(x-v_0t) = \rho u_{tt}$, $u(x,0) = u_t(x,0) = 0$.

12. $v_x+Li_t+Ri = 0$, $i_x+Cv_t+Gv = 0$, $v(x,0) = \varphi(x)$, $i(x,0) = \varphi_1(x)$.

13. $v_x+Li_t = 0$, $i_x+Cv_t = 0$, $v(x,0) = \varphi(x)$, $i(x,0) = \varphi_1(x)$, $-v(0,t) = R_0i(0,t)$, $c_0v_t(l,t) = i(l,t)$.

14. $v_x+Li_t = 0$, $i_x+Cv_t = 0$, $v(x,0) = \varphi(x)$, $i(x,0) = \varphi_1(x)$, $-v(0,t) = L_0^{(1)}i_t(0,t)$, $v(l,t)-E(t) = L_0^{(2)}i_t(l,t)$.

15. $v(x,t) = \begin{cases} v_1(x,t), & x<0 \\ v_2(x,t), & x>0 \end{cases}$ $\quad i(x,t) = \begin{cases} i_1(x,t), & x<0 \\ i_2(x,t), & x>0 \end{cases}$
$(v_k)_x+L_k(i_k)_t+R_ki_k = 0$, $(i_k)_x+C_k(v_x)_t = 0$, $v(x,0) = \varphi(x)$,
$i(x,0) = \varphi_1(x)$, $i_1(0,t) = i_2(0,t)$,
$v_{2t}(0,t)-v_{1t}(0,t) = (1/c_0)i_1(0,t) \quad (k = 1, 2)$.

For the current $i(x, t)$:

$$(i_k)_{xx} = C_k L_k (i_k)_{tt} + C_k R_k i_k, \qquad i_1(0, t) = i_2(0, t),$$

$$\frac{1}{C_1} i_{1x}(0, t) - \frac{1}{C_2} i_{2x}(0, t) = \frac{1}{C_0} i_1(0, t), \qquad i_k(x, 0) = \varphi(x),$$

$$i_{kt}(x, 0) = \frac{R_k \varphi(x) - \varphi_1'(x)}{L_k}.$$

**16.** $k u_{xx} - h[u - \varphi(t)] = c\rho u_t, \quad u(x, 0) = f(x), \quad u(0, t) = f_1(t),$
$k u_x(l, t) = q(t)$.

**17.** $k u_{xx} - hu + QI^2 = c\rho u_t,\ u(x, 0) = f(x),\ k u_x(0, t) = C_1 u_t(0, t),$
$k u_x(l, t) = C_2 u_t(l, t)$, where $C_1$, $C_2$ are the thermal capacities of the holders.

**18.** $\partial/\partial x\,(D u_x) - \partial/\partial x\,(vu) = c u_t$.

**19.** **(a)** $\partial/\partial x\,(D u_x) - \beta u = c u_t$;

**(b)** $\partial/\partial x\,(D u_x) + \beta u = c u_t$.

**20.** $u(x,t) = \begin{cases} u_1(x, t), & x < 0 \\ u_2(x, t), & x > 0 \end{cases} \qquad a_i^2 (u_i)_{xx} = (u_i)_t \qquad (i = 1, 2),$

$u(x, 0) = \varphi(x), \qquad u_1(0, t) = u_2(0, t)$;

**(a)** $k_2 u_{2x}(0, t) - k_1 u_{1x}(0, t) = 0$;

**(b)** $k_2 u_{2x}(0, t) - k_1 u_{1x}(0, t) = C_0 u_t(0, t)$.

**21.** $\partial/\partial x\,(k u_x) = c\rho u_t,\ u(x, 0) = 0, \quad u(vt, t) = \varphi(t)$.

**22.** $\partial/\partial x\,(k u_x) + Q\delta(x - v_0 t) = c\rho u_t, \quad u(x, 0) = \varphi(x)$.

**23.** $a^2 u_{\theta\theta} - h(u - u_0) = u_t,\ \ u(\theta, 0) = \varphi(\theta),\ \ u(\theta + 2\pi, t) = u(\theta, t),$
$\theta$ is the polar angle, $a^2 = k/c\rho R^2$, $R$ is the radius of the ring.

**24.** $$\frac{\partial}{\partial t}\boldsymbol{E} = \frac{c^2}{4\pi\sigma\mu}\frac{\partial^2 \boldsymbol{E}}{\partial \xi^2}, \qquad \frac{\partial \boldsymbol{H}}{\partial t} = \frac{c^2}{4\pi\sigma\mu}\frac{\partial^2 \boldsymbol{H}}{\partial \xi^2},$$

where $c$ is the velocity of light, $\sigma$ is the conductivity of the medium, $\mu$ is the magnetic permeability, $\xi$ is the distance measured from a fixed plane, $\boldsymbol{E} = \boldsymbol{E}(\xi, t)$, $\boldsymbol{H} = \boldsymbol{H}(\xi, t)$.

**25.** **(a)** $\Delta u = -4\pi\rho$;

**(b)** $\Delta u = 0$.

## CHAPTER 3

**1.** See Fig. B.1. *Hint*: use D'Alembert's formula.

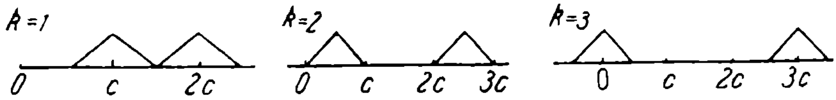



*Fig. B.1*

**2.** See Fig. B.2.

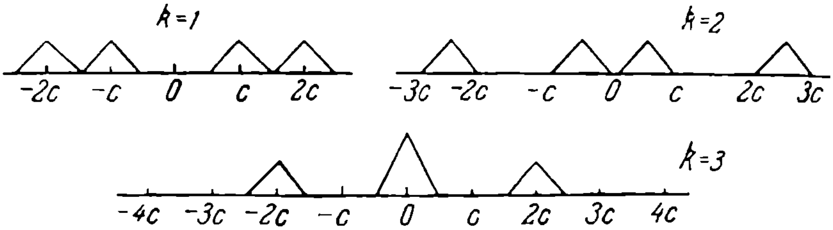



*Fig. B.2*

**3.** $u(x, t) = 1/2a\{F(x+at)-F(x-at)\}$,

where $F(z) = \begin{cases} 0, & z \leqslant -c \\ v_0(z+c), & -c \leqslant z \leqslant c \\ 2v_0 c, & z \geqslant c \end{cases}$

**4.** See Fig. B.3.

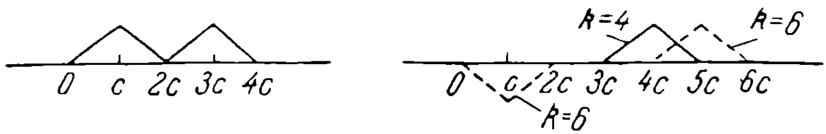



*Fig. B.3*

**5.** $u(x, t) = 1/2a\{F(x+at)-F(x-at)\}$,

where $F(z) = p/\rho\{\eta(z-x_0)-\eta(z+x_0)\}$.

*Hint*: solve the problem: $a^2 u_{xx} = u_{tt}$, $u(0, t) = 0$, $u(x, 0) = 0$, $u_t(x, 0) = (p/\rho)\delta(x-x_0)$, $0 \leqslant x < \infty$.

**6.** For $-\infty < x < 0$ we have

$$u_1(x, t) = f(t-x/a_1) + \frac{\sqrt{\rho_1 E_1}-\sqrt{\rho_2 E_2}}{\sqrt{\rho_1 E_1}+\sqrt{\rho_2 E_2}} f(t+x/a_1).$$

Refracted wave: $u_2(x, t) = \dfrac{2\sqrt{\rho_1 E_1}}{\sqrt{\rho_1 E_1}+\sqrt{\rho_2 E_2}} f(t-x/a_2)$.

Reflected wave: $u_{\text{refl}} = \dfrac{\sqrt{\rho_1 E_1}-\sqrt{\rho_2 E_2}}{\sqrt{\rho_1 E_1}+\sqrt{\rho_2 E_2}} f(t+x/a_1)$.

$u_{\text{refl}}$ absent for $\rho_1 E_1 = \rho_2 E_2$.

7. $v(x, 0) = E_0\,e^{-\sqrt{GR}\,x}, \quad i(x, 0) = E_0\sqrt{G/L}\,e^{-\sqrt{GR}\,x}$,
$v(x, t) = E_0\,e^{-\sqrt{GR}\,x}\eta(x-at), \quad 0 < x, \quad t < \infty$,
$i(x, t) = E_0\sqrt{G/L}\,e^{-\sqrt{GR}\,x}\eta(x-at)$.

8. $u(x, t_1) \equiv 0, \quad u(x, t_2) = -u(x, 0), \quad u(x, t_4) \equiv u(x, 0)$.

9. $$u_1(x, t) = \begin{cases} \dfrac{a^2-v_0^2}{2aT_0}\displaystyle\int_0^{\frac{x+at}{a+v_0}} F(\xi)\,d\xi, & -at < x < v_0 t, \\ 0, & x > at, \end{cases}$$

$$u_2(x, t) = \frac{a^2-v_0^2}{2aT_0}\begin{cases} \displaystyle\int_0^{\frac{at-x}{a-v_0}} F(\xi)\,d\xi, & v_0 t < x < at, \\ 0, & x > at, \end{cases}$$

$T_0$ is the initial tension in the string.

10. $u(x, t) = \eta(t-x/a) f(t-x/a), \quad a = 1/\sqrt{LC}$.

11. $I(x, t) = \eta(t-x/a) V\sqrt{C/L}\,e^{-\alpha(t-x/a)}$,
where $a = 1/\sqrt{LC}, \quad \alpha = (1/C_0)\sqrt{C/L}$.

12. $S(M, t) = S_0\dfrac{\partial}{\partial t}\left\{\dfrac{\sigma_t}{4\pi a^2 t}\right\} = \dfrac{S_0}{4\pi a^2}\dfrac{\partial}{\partial t}\left(\dfrac{\sigma_t}{t}\right)$.

13. Only equations of the hyperbolic type of the form
$a_{11}u_{xx}+2a_{12}u_{xt}+a_{22}u_{tt} = 0$.
The velocity $a$ is determined from the equation
$a_{22}a^2-2a_{12}a+a_{11} = 0$.

14. Only equations of the hyperbolic type with coefficients satisfying the following relationships:

$a_{22}a^2-2a_{12}a+a_{11}=0, \quad b_1-b_2a-2\mu(a_{12}-a_{22}a)=0,$
$a_{22}a^2-\mu b_2+c=0.$

## CHAPTER 4

1. $$u(x,t)=\frac{-2hl^2}{\pi^2x_0(l-x_0)}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{\pi n}{l}x_0\sin\frac{\pi n}{l}x\cos\frac{\pi na}{l}t,$$

where $h=\frac{1}{lT}F_0x_0(l-x_0)$.

2. $$u(x,t)=\frac{2P}{\pi a\rho}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{\pi n}{l}x_0\sin\frac{\pi n}{l}x\sin\frac{\pi an}{l}t.$$

*Hint*: $u(x,0)=0, \quad u_t(x,0)=\frac{P}{\rho}\delta(x-x_0)$.

3. $$u(x,t)=\frac{8lF_0}{\pi^2ES}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\sin\frac{2n+1}{2l}\pi x\cos\frac{2n+1}{2l}a\pi t.$$

*Hint*: $u(x,0)=\frac{F_0}{ES}x, \quad u_t(x,0)=0$.

4. $$u(x,t)=\frac{2P}{a\rho}\sum_{n=1}^{\infty}\frac{\cos\frac{\mu_n}{l}x\sin\frac{\mu_n}{l}at}{\mu_n\left\{1+\frac{h}{l\left(h^2+\frac{\mu_n^2}{l^2}\right)}\right\}},$$

where $\mu_n$ are the positive roots of the equation $\mu\tan\mu=hl$.

5. $$u(r,t)=\sum_{n=1}^{\infty}C_n e^{-a^2\lambda_n t}\frac{1}{r}\sin\frac{\mu_n}{R}r, \quad \text{where} \quad \mu_n=\sqrt{\lambda_n}R,$$

$$C_n=\frac{2}{R}\frac{R^2\mu_n^2+(Rh-1)^2l^2}{R^2\mu_n^2+(Rh-1)Rhl^2}\int_0^R rf(r)\sin\frac{\mu_n}{R}r\,dr,$$

$\mu_n$ are the positive roots of the equation $\tan\mu=\frac{\mu}{1-hR}$.

*Hint*: $u=v/r$ and $a^2v_{rr}=v_{tt}$.

6. $u(r,t)=\sum_{n=1}^{\infty} C_n \mathrm{e}^{-a^2\lambda_n t}\frac{1}{r}\Phi_n(r),$

where $\Phi_n(r)=(1-h_1R_1)\sin\sqrt{\lambda_n}\,r+\sqrt{\lambda_n}\cos\sqrt{\lambda_n}\,r,$

$$C_n=\frac{1}{\|\Phi_n\|^2}\int_{R_1}^{R_2} rf(r)\Phi_n(r)\mathrm{d}r,\qquad \|\Phi_n\|^2=\int_{R_1}^{R_2}\Phi_n^2(r)\mathrm{d}r,$$

$\lambda_n$ are the positive roots of the equation

$$\tan\sqrt{\lambda}\,(R_2-R_1)=\frac{R_2(1-h_1R_1)-(1-h_2R_2)}{R_2\lambda+(1-h_1R_1)(1-h_2R_2)}.$$

7. (a) $R_{cr}=\pi a/\sqrt{\beta}$;

(b) $R_{cr}=0$;

(c) $R_{cr}$ is equal to the smallest positive root of the equation

$$(1-hR)\tan\frac{\sqrt{\beta}}{a}R=\frac{\sqrt{\beta}}{a}R.$$

8. For boundary conditions of type I:

$$\lambda_{n,p}=\pi^2\left(\frac{n^2}{l^2}+\frac{p^2}{k^2}\right),\qquad \Phi_{n,p}(x,y)=\sin\frac{\pi n}{l}x\sin\frac{\pi p}{k}y$$

$(n, p = 1, 2, \ldots)$.

In the case of $(l=k)$, $\lambda_{n,p}=\pi^2/l^2(n^2+p^2)$,

$\lambda_{1,2}=\lambda_{2,1}=5\pi^2/l^2$.

but $\Phi_{1,2}(x,y)=\sin\frac{2\pi}{l}x\sin\frac{\pi}{l}y\neq\Phi_{2,1}=\sin\frac{\pi}{l}x\sin\frac{2\pi}{l}y.$

Therefore, to a single eigenvalue $\lambda=5\pi^2/l^2$ there correspond two linearly independent eigenfunctions.

For boundary conditions of type II:

$$\lambda_{n,p}=\pi^2\left(\frac{n^2}{l^2}+\frac{p^2}{k^2}\right),\quad \Phi_{n,p}(x,y)=\cos\frac{\pi n}{l}x\cos\frac{\pi p}{k}y$$

$(n, p = 0, 1, 2, \ldots)$;

for boundary conditions of type III:

$\lambda_{n,p}=\mu_n+\alpha_p,$

$$\Phi_{n,p}(x,y)=(\sqrt{\mu_n}\cos\sqrt{\mu_n}\,x+h_1\sin\sqrt{\mu_n}\,x)$$
$$\times(\sqrt{\alpha_p}\cos\sqrt{\alpha_p}\,y+h_3\sin\sqrt{\alpha_p}\,y),$$

$\mu_n$ and $\alpha_p$ are positive roots of the equations

$$\tan\sqrt{\mu}\,l = \frac{-(h_1+h_2)\sqrt{\mu}}{h_1h_2-\mu}, \qquad \tan\sqrt{\alpha}\,k = \frac{-(h_3+h_4)\sqrt{\alpha}}{h_3h_4-\alpha}.$$

**9.** For boundary conditions of type I:

$$\lambda_{n,p,q} = \pi^2\left(\frac{n^2}{l^2}+\frac{p^2}{k^2}+\frac{q^2}{m^2}\right),$$

$$\Phi_{n,p,q}(x, y, z) = \sin\frac{\pi n}{l}x \sin\frac{\pi p}{k}y \sin\frac{\pi q}{m}z \quad (n, p, q = 1, 2, \ldots);$$

for boundary conditions of type II:

$$\lambda_{n,p,q} = \pi^2\left(\frac{n^2}{l^2}+\frac{p^2}{k^2}+\frac{q^2}{m^2}\right),$$

$$\Phi_{n,p,q}(x, y, z) = \cos\frac{\pi n}{l}x \cos\frac{\pi p}{k}y \cos\frac{\pi q}{m}z \quad (n,p,q = 0, 1, 2, \ldots);$$

for boundary conditions of type III:

$$\lambda_{n,p,q} = \mu_n+\alpha_p+\beta_q,$$

$$\begin{aligned}\Phi_{n,p,q}(x, y, z) = {} & (\sqrt{\mu_n}\cos\sqrt{\mu_n}\,x+h_1\sin\sqrt{\mu_n}\,x)\\ & \times(\sqrt{\alpha_p}\cos\sqrt{\alpha_p}\,y+h_3\sin\sqrt{\alpha_p}\,y)\\ & \times(\sqrt{\beta_q}\cos\sqrt{\beta_q}\,z+h_5\sin\sqrt{\beta_q}\,z),\end{aligned}$$

$\mu_n, \alpha_p, \beta_q$ are the positive roots of the following equations:

$$\tan\sqrt{\mu}\,l = \frac{-(h_1+h_2)\sqrt{\mu}}{h_1h_2-\mu}, \qquad \tan\sqrt{\alpha}\,k = \frac{-(h_3+h_4)\sqrt{\alpha}}{h_3h_4-\alpha},$$

$$\tan\sqrt{\beta}\,m = \frac{-(h_5+h_6)\sqrt{\beta}}{h_5h_6-\beta}.$$

**10.** **(a)** $\omega_{n,p,q} = a\sqrt{\lambda_{n,p,q}} = a\pi\sqrt{\frac{n^2}{l^2}+\frac{p^2}{k^2}+\frac{q^2}{m^2}}$

$(n, p, q = 1, 2, \ldots)$;

**(b)** $\omega_{n,p} = a\mu_{n,p}\,1/R$, where $\mu_{n,p}$ is the root of number $p$ of the equation $J'_{n+1/2}(\mu)-\frac{1}{2\mu}J_{n+1/2}(\mu) = 0$, $R$ is the radius of the sphere $n = 0, 1, 2, \ldots, p = 1, 2, \ldots$, where $J_k(z)$ are Bessel functions of order $k$ (see Chapter 11)

**11.** $$u(x, t) = \frac{8l}{a\pi^2}\sum_{n=0}^{\infty}\left\{\frac{2lg}{\pi a(2n+1)^3}\cos\frac{2n+1}{2l}at+\frac{v_0}{(2n+1)^2}\right.$$

$$\times \sin \frac{2n+1}{2l} at \Big\} \sin \frac{2n+1}{2l} \pi x - \frac{g}{a^2}\left(\frac{x^2}{2} - lx\right).$$

**12.** $$u(x, t) = \frac{F_0}{ES} x - \frac{8F_0 l}{ES\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \sin \frac{2n+1}{2l} \pi x$$

$$\times \cos \frac{2n+1}{2l} \pi a t.$$

*Hint for Problems* 12–23: the solution should be sought in the form of the sum $u = v(x)+w(x, t)$ where $v(x)$ satisfies the equation and the boundary conditions of the inhomogeneous boundary-value problem under consideration, whilst $w$ is the solution of the corresponding homogeneous boundary-value problem; $v(x)$ describes the steady state, $w$ the departure from this state.

**13.** $$u(x, t) = u_3 + v(x) + \sum_{n=1}^{\infty} C_n e^{-(a^2\lambda_n + h)t} \sin \frac{\pi n}{l} x, \quad \text{where}$$

$$v(x) = \frac{1}{\sinh \dfrac{l\sqrt{h}}{a}} \left[ (u_1 - u_3) \sinh \frac{\sqrt{h}}{a}(l-x) + (u_2 - u_3) \sinh \frac{\sqrt{h}}{a} x \right],$$

$$\lambda_n = \frac{\pi^2 n^2}{l^2}, \qquad C_n = \frac{2}{l} \int_0^l \{f(\xi) - v(\xi) - u_3\} \sin \frac{\pi n}{l} \xi \, d\xi.$$

**14.** $$u(x, t) = u_3 + v(x) + \sum_{n=1}^{\infty} C_n e^{-(a^2\lambda_n + h_3)t}$$

$$\times (h_1 \sin \sqrt{\lambda_n}\, x + \sqrt{\lambda_n} \cos \sqrt{\lambda_n}\, x),$$

where $v(x) = A_1 e^{\frac{\sqrt{h_3}}{a} x} + A_2 e^{-\frac{\sqrt{h_3}}{a} x}$. The coefficients $A_1$ and $A_2$ are determined from the following equations:

$$A_1 - A_2 = \frac{h_1(u_3 - u_1)}{\sqrt{h_3} - a h_1} a,$$

$$A_1\left(h_2 + \frac{\sqrt{h_3}}{a}\right) e^{\frac{\sqrt{h_3}}{a} l} + A_2\left(h_2 - \frac{\sqrt{h_3}}{a}\right) e^{-\frac{\sqrt{h_3}}{a} l} = h_2 u_2,$$

$$C_n = \frac{1}{\|\Phi_n\|^2} \int_0^l \{\varphi(\xi) - u_3 - v(\xi)\} \Phi_n(\xi)\, d\xi,$$

$$\Phi_n(x) = h_1 \sin\sqrt{\lambda_n} x + \sqrt{\lambda_n} \cos\sqrt{\lambda_n} x.$$

$\lambda_n$ are the positive roots of the equation $\tan\sqrt{\lambda} l = \dfrac{h_1+h_2}{\lambda - h_1 h_2}\sqrt{\lambda}$.

**15.** $Q(t) = S\int_0^l u(x, t)\, dx$, where $S$ is the area of the transverse cross-section of a cylinder,

$$u(x, t) = u_0 - \sum_{n=0}^{\infty} \frac{4u_0}{\pi(2n+1)} e^{-a^2\lambda_n t} \sin\frac{2n+1}{2l}\pi x,$$

$$\lambda_n = \frac{\pi^2}{4l^2}(2n+1)^2 \quad \text{or} \quad Q(t) = -D\int_0^t u_x(0, \tau)\, d\tau.$$

**16.** $Q(t) = S\int_0^l u(x, t)\, dx \quad (a^2 u_{xx} - \beta u = u_t)$,

where $u(x, t) = v(x) + \sum_{n=0}^{\infty} C_n e^{-a^2\lambda_n t} \sin\dfrac{2n+1}{2l}\pi x\, e^{-\beta t}$,

$$v(x) = \frac{a u_0}{\sqrt{\beta}\cosh\frac{\sqrt{\beta}}{a} l} \cosh\frac{\sqrt{\beta}}{a}(l-x),$$

$$C_n = \frac{-2}{l}\int_0^l v(\xi) \sin\frac{2n+1}{2l}\pi\xi\, d\xi.$$

**17.** $v(x, t) = E_0 - \dfrac{4(E_0 - v_0)}{\pi}\sum_{n=0}^{\infty} \dfrac{(-1)^n}{2n+1} e^{-a^2\lambda_n t} \cos\dfrac{2n+1}{2l}\pi x$,

where $\lambda_n = \pi^2(2n+1)^2/4l^2$, $\quad a^2 = 1/RC$.

**18.** $$v(x, t) = \frac{E_0 R(l-x)}{R_0 + Rl} + 2E_0 R^2 \sum_{n=1}^{\infty} e^{-a^2\lambda_n t} \times \frac{\sin\sqrt{\lambda_n}(l-x)}{\sqrt{\lambda_n}[R(R_0+Rl) + lR_0^2\lambda_n]\cos\sqrt{\lambda_n} l},$$

where $a^2 = 1/RC$, $\lambda_n$ are the positive roots of the equation $R\tan\sqrt{\lambda_n}\, l = -R_0\sqrt{\lambda}$.

**19.** $$H(x, t) = H_0 - \frac{4H_0}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\, e^{-a^2\lambda_n t}\sin\frac{2n+1}{2l}\pi x,$$

where $a^2 = c^2/4\pi\sigma\mu$, $\quad \lambda_n = \pi^2(2n+1)^2/4l^2$.

**20.** $$u(x, t) = \frac{2a^2Q}{kl}\sum_{n=1}^{\infty}\left\{\frac{1}{\lambda_n}\Phi_n(x_0)\,(1-e^{-a^2\lambda_n t})\right\}\Phi_n(x),$$

where $\lambda_n = \dfrac{\pi^2}{4l^2}(2n+1)^2$, $\quad \Phi_n(x) = \cos\dfrac{2n+1}{2l}\pi x$,

$k u_{xx} + Q\delta(x-x_0) = c\rho u_t$,
$k$ is the thermal conductivity.

**21.** $$u(x, t) = u_0 + \frac{ql}{k}\left(1-\frac{x}{l}\right) - \sum_{n=0}^{\infty}\frac{8l}{\pi^2}\left[\frac{q}{k} + (-1)^n\frac{2n+1}{2l}\pi u_0\right]$$

$$\times e^{-a^2\lambda_n t}\frac{1}{(2n+1)^2}\cos\frac{2n+1}{2l}\pi x,$$

where $\lambda_n = \dfrac{(2n+1)^2}{4l^2}$, $k$ is the thermal conductivity.

**22.** $$u(x, t) = v(x) + \sum_{n=1}^{\infty} C_n\, e^{-a^2\lambda_n t}\Phi_n(x),$$

where $v(x) = \dfrac{-Q}{2k}x + C_0(x+h)$, $\quad C_0 = \dfrac{Ql}{k}\left(1+\dfrac{hl}{2}\right)\dfrac{1}{1+hl+h^2}$,

$$C_n = \frac{-1}{\|\Phi_n\|^2}\int_0^l v(\xi)\,\Phi_n(\xi)\,d\xi, \quad \Phi_n(x) = \sqrt{\lambda_n}\cos\sqrt{\lambda_n}\, x$$
$$+ h\sin\sqrt{\lambda_n}\, x,$$

$\lambda_n$ positive roots of the equation $\tan\sqrt{\lambda}\, l = \dfrac{2h\sqrt{\lambda}}{\lambda - h^2}$.

**23.** $$u(r, t) = u_1 + 2(u_1-u_0)hR^2\sum_{n=1}^{\infty}(-1)^n\frac{C_n}{r}\, e^{-a^2\frac{\mu_n^2}{l^2}t}\sin\frac{\mu_n}{R}r,$$

where $C_n = \dfrac{\sqrt{\mu_n^2 + (hR-1)^2}}{\mu_n(\mu_n^2 + h^2R^2 - hR)}$, $\mu_n$ are the positive roots of the

equation $\tan\mu = -\mu/(Rh-1)$, $h$ is the heat transfer coefficient in the boundary condition $u_r(R,t)+h[u(R,t)-u_1] = 0$.

**24.** $\Delta u = 0, \quad u_r(R,\varphi) = \frac{Q}{\pi k R}, \quad u_\varphi(r,0) = \frac{Qr}{2kR}, \quad u_\varphi(r,\pi) = \frac{-Qr}{2kR},$

$$u(r,\varphi) = \frac{Qr}{2kR}\sin\varphi + \frac{Q}{2\pi k}\sum_{n=0}^{\infty}\left(\frac{r}{R}\right)^{2n+1}\frac{\cos(2n+1)\varphi}{n(n+1)}.$$

*Hint*: first find the solution of the Laplace equation of the form $rv(\varphi)$ satisfying only the conditions: $u_\varphi(r,0) = Qr/2kR$, $u_\varphi(r,\pi) = -Qr/2kR$, and the deflection $w(r,\varphi)$ from it. Then, $u = rv(\varphi)+w(r,\varphi)$.

**25.** $a^2\theta_{xx} = \theta_{tt}, \quad \theta(x,0) = \alpha x/l, \quad \theta_t(x,0) = 0,$

$$\theta(0,t) = 0, \quad \theta_x(l,t) = -\frac{I_0}{GI}\theta_{tt}(l,t), \quad a^2 = \sqrt{G/\rho}.$$

$$\theta(x,t) = 4\alpha\sum_{n=1}^{\infty}\frac{(I_1-I_0\mu_n^2)\sin\frac{\mu_n}{l}x}{I_0\mu_n^2(2\mu_n-\sin 2\mu_n)}\cos\frac{a\mu_n}{l}t,$$

where $I$ is the polar moment of inertia of the transverse cross section of the rod, $G$ is the shear modulus, $I_1$ is the moment of inertia of the rod, $\rho$ is the linear density of the rod and $\mu_n$ are the positive roots of the equation $\tan\mu = I_1/I_0\mu$.

**26.** **(a)** $$u(x,t) = \frac{4a\Phi_0 l^2}{\pi^2 T}\sum_{n=1}^{\infty}\frac{\omega l\sin\frac{2n+1}{l}a\pi t-(2n+1)\pi a\sin\omega t}{(2n+1)^2[\omega^2 l^2-(2n+1)^2\pi^2 a^2]}\times\sin\frac{2n+1}{l}\pi x;$$

**(b)** Replace $\sin\omega t$ by $\cos\omega t$ in the last formula ($\omega \neq k\pi/l$, $k = 1, 2, 3, \ldots$).

**27.** $$u(x,t) = \frac{2F_0 a l}{\pi T}\sum_{n=1}^{\infty}\frac{\omega l\sin\frac{\pi n}{l}at-\pi a n\sin\omega t}{(\omega^2 l^2-\pi^2 a^2 n^2)n}\sin\frac{\pi n}{l}t\sin\frac{\pi n}{l}x$$

($\omega \neq \pi n/l$, $n = 1, 2, \ldots$) and similarly for $F_0\cos\omega t$.

**28.** $$u(x,t)=\frac{-a^2}{k}\sin\frac{\pi x}{l}\int_0^t e^{-\frac{\pi^2a^2}{l^2}(t-\tau)}\Phi(\tau)\,d\tau+\sum_{n=1}^{\infty}C_n e^{\frac{-\pi^2n^2a^2}{l^2}t}\sin\frac{\pi n}{l}x,$$

where $k$ is the thermal conductivity, $C_n=\frac{2}{l}\int_0^l f(\xi)\sin\frac{\pi n}{l}\xi\,d\xi$.

**29.** $$u(x,t)=\frac{2Al}{c\rho}e^{-ht}\sum_{n=1}^{\infty}\left(\sin\frac{\pi n v_0}{l}t-\frac{v_0 l}{\pi n}\cos\frac{\pi n v_0}{l}t+\frac{v_0 l}{\pi n}\right)\times\frac{\sin\frac{\pi n}{l}x}{l^2v_0^2+\pi^2n^2a^2}.$$

*Hint*: the equation for $u(x,t)$ is of the form

$$a^2u_{xx}-hu+\frac{A}{c\rho}\delta(x-v_0t)=u_t,\quad 0<t<\frac{l}{v_0}.$$

**30.** **(a)** For $\omega\neq\frac{2n+1}{2l}\pi a\quad(n=0,1,2,\ldots)$

$$u(x,t)=v(x)\sin\omega t+\sum_{n=0}^{\infty}C_n\sin\frac{2n+1}{2l}\pi at\sin\frac{2n+1}{2l}x,$$

$$v(x)=\frac{Aa}{ES}\frac{\sin\frac{\omega}{a}x}{\cos\frac{\omega}{a}l},\qquad C_n=\frac{-4\omega}{\pi a(2n+1)}\int_0^l v(\xi)\sin\frac{2n+1}{2l}\pi\xi\,d\xi;$$

**(b)** for $\omega=\frac{2n_0+1}{2l}\pi a$

$$u(x,t)=v_1(x)\sin\omega t+v_2(x)t\cos\omega t+\sum_{\substack{n=0\\ n\neq n_0}}^{\infty}C_n\sin\frac{2n+1}{2l}\pi at\sin\frac{2n+1}{2l}\pi x,$$

where $$C_n=\frac{-4}{\pi a(2n+1)}\int_0^l[\omega v_1(\xi)+v_2(\xi)]\sin\frac{2n+1}{2l}\pi\xi\,d\xi,$$

$$v_1(x)=\frac{A}{lES}(-1)^{n_0+1}\left\{\frac{x}{2}\cos\frac{\omega}{a}x+\frac{5a}{8\omega}\sin\frac{\omega}{a}x+\frac{3a}{8\omega}\sin\frac{3\omega}{a}x\right\},$$

$$v_2(x) = \frac{2aA}{lES}(-1)^{n_0} \sin\frac{\omega}{a}x$$

and similarly for $F = A\cos\omega t$.

*Hint*: the solution should be sought for (a) in the form $u = v(x) \sin\omega t + w(x, t)$; in case (b) in the form $u = v_1(x)\sin\omega t + v_2(x)t\cos\omega t + w(x, t)$, where $v(x)\sin\omega t$ (correspondingly, $v_1(x)\sin\omega t + v_2(x)t\cos\omega t$) satisfies the equation and the boundary conditions of the problem.

**31.** $$u(r, t) = F_1(r) + tF_2(r) - \frac{2qR^2}{k_1 r}\sum_{n=1}^{\infty}\frac{1}{\mu_n^2\cos\mu_n} e^{-\frac{a^2\mu_n^2}{R^2}t}\sin\frac{\mu_n}{R}r,$$

$\mu_n$ are the positive roots of the equation $\tan\mu = \mu$,

$$F_1(r) = u_0 + \frac{qR}{k_1}\frac{3R^2 - 5r^2}{10R^2}, \qquad F_2(r) = \frac{3qa^2}{k_1 R}.$$

*Hint*: determine the solution of the problem $a^2 v_{rr} = v_t$, $v(r, 0) = u_0 r$, $k_1[Rv_r(R, t) - v(R, t)] = q$ $(u = v/r)$ in the form $v = f_1(r) + tf_2(r) + w(r, t)$, where $(f_1 + tf_2)/r$ is the steady state satisfying the equation and the boundary conditions and $w/r$ is the deviation from it; $w$ is the solution of the homogeneous boundary-value problem.

**32.** $u(x, t)$ is the solution of the problem

$$\partial/\partial x\,[k(x)u_x] = \rho(x)u_t, \quad u(0, t) = u(l, t) = 0, \quad u(x, 0) = f(x),$$

where $k(x) = \begin{cases} k_1, & 0 < x < x_0, \\ k_2, & x_0 < x < l, \end{cases} \quad \rho(x) = \begin{cases} \rho_1, & 0 < x < x_0, \\ \rho_2, & x_0 < x < l, \end{cases}$

or $u(x, t) = \begin{cases} u_1(x, t), & 0 \leqslant x \leqslant x_0, \\ u_2(x, t), & x_0 \leqslant x \leqslant l, \end{cases}$

$$a_1^2(u_1)_{xx} = (u_1)_t, \quad a_2^2(u_2)_{xx} = (u_2)_t, \quad a_i^2 = \frac{k_i}{\rho_i} \quad (i = 1, 2),$$

$$u_1(0, t) = 0, \quad u_2(l, t) = 0, \quad u_1(x_0, t) = u_2(x_0, t),$$

$$k_1 u_{1x}(x_0, t) = k_2 u_{2x}(x_0, t), \qquad u(x, 0) = f(x),$$

$$u(x, t) = \sum_{n=1}^{\infty} C_n e^{-\mu_n^2}\Phi_n(x),$$

$$\text{where } \Phi_n(x) = \begin{cases} \dfrac{1}{\sin\dfrac{\mu_n}{a_1}x_0}\sin\dfrac{\mu_n}{a_1}x_1, & 0 \leqslant x \leqslant x_0, \\[2ex] \dfrac{1}{\sin\dfrac{\mu_n}{a_2}(l-x_0)}\sin\dfrac{\mu_n}{a_2}(l-x_0), & x_0 \leqslant x \leqslant l, \end{cases}$$

$$C_n = \frac{1}{\|\Phi_n\|^2}\int_0^l f(x)\rho(x)\Phi_n(x)\,dx,$$

$\mu_n$ are the positive roots of the equation $\dfrac{k_1}{a_1}\cot\dfrac{\mu}{a_1}x_0 = \dfrac{k_2}{a_2}$ $\cot\dfrac{\mu}{a_2}(x_0-l)$. The eigenfunctions $\Phi_n(x)$ are orthogonal with the weight $\rho(x)$ in the range $[0, l]$.

**33.** $u(x, t)$ is the solution of the problem

$a^2u_{xx} = [1+(C_0/C)\delta(x-x_0)]u_t,$

$u(0, t) = u(l, t) = 0, \quad u(x, 0) = f(x),$

$$\text{or } u(x, t) = \begin{cases} u_1(x, t), & 0 < x < x_0, \\ u_2(x, t), & x_0 < x < l, \end{cases}$$

$a^2(u_i)_{xx} = (u_i)_t, \quad (i = 1, 2), \quad u_1(0, t) = 0 = u_2(l, t),$

$u_1(x_0, t) = u_2(x_0, t), \quad u(x, 0) = f(x),$

$ku_{2x}(x_0, t)-ku_{1x}(x_0, t) = C_0u_t(x_0, t),$

$$u(x, t) = \sum_{n=1}^{\infty} C_n e^{-a^2\mu_n^2 t}\Phi_n(x),$$

$$\text{where } C_n = \frac{1}{\|\Phi_n\|^2}\int_0^l \rho(x)f(x)\Phi_n(x)\,dx,$$

$$\Phi_n(x) = \begin{cases} \dfrac{\sin\mu_n x}{\sin\mu_n x_0}, & 0 \leqslant x \leqslant x_0, \\[2ex] \dfrac{\sin\mu_n(l-x)}{\sin\mu_n(l-x_0)}, & x_0 \leqslant x \leqslant l, \end{cases}$$

$\mu_n$ are the positive roots of the equation $\cot\mu x_0-\cot\mu(l-x_0)$ $= \dfrac{C_0}{C\rho}\mu.$

Eigenfunctions $\Phi_n(x)$ are orthogonal in the range $[0, l]$ with the weight $\rho(x) = 1+\frac{C_0}{C}\delta(x-x_0)$.

**34.** $u(x, t) = E_0+2lE_0C\sum\limits_{n=1}^{\infty} e^{-\frac{\mu_n^2 t}{l^2RC}}$

$$\times \frac{C_0\mu_n \sin \mu_n\left(1-\frac{x}{l}\right)-Cl\cos \mu_n\left(1-\frac{x}{l}\right)}{(lCC_0+l^2C^2+C_0^2\mu_n^2)\mu_n \sin \mu_n},$$

where $\mu_n$ are the positive roots of the equation $\mu \tan \mu = Cl/C_0$. *Hint*: the eigenfunctions of the problem are orthogonal in the range $[0, l]$ with the weight $\rho(x) = 1+\frac{C_0}{C}\delta(x-l)$.

**35.** **(a)** $u(r, \varphi) = \frac{A}{R} r\cos\varphi = \frac{A}{R}x$;

**(b)** $u = A+\frac{B}{R}y$;

**(c)** $u = Axy$;

**(d)** $u = \frac{A+B}{2}+\frac{B-A}{2R^2}(x^2-y^2)$.

*Hint for Problems* 35–41. See Example 1 of Section 6.3.

**36.** Problem **(a)** is incorrectly formulated since the necessary condition $\int\limits_C \frac{\partial u}{\partial n}\,ds = 0$ is not satisfied;

**(b)** $u = ARx+D$;

**(c)** $u = \frac{A}{2}R(x^2-y^2)$;

**(d)** $u = \left(A+\frac{3}{4}B\right)y-\frac{B}{12R^2}[3(x^2+y^2)y-4y^3]+D$,

where $D$ is an arbitrary constant.

**37.** $u = u(r) = u_1+(u_2-u_1)\frac{\ln r/R_1}{\ln R_2/R_1}$. The capacitance per unit length of the cylindrical capacitor is $C = 1/(\ln R_2-\ln R_1)$.

*Hint*: the capacitance $C$ of the conductor bounded by the surface $S$ in three dimensions is $C = \frac{-1}{4\pi u_0}\int_S \frac{\partial u}{\partial n}\,d\sigma$, in two dimensions $C = \frac{-1}{2\pi u_0}\int_L \frac{\partial u}{\partial n}\,ds$, where $L$ is the contour, $u_0$ the potential of the conductor, $\partial u/\partial n = E_n$ is the normal component of the electric field stress tensor.

**38.** $$C = \frac{\varepsilon_1}{u_0\left[\frac{1}{a}+\frac{1}{c}-\frac{\varepsilon_1}{\varepsilon_2}\left(\frac{1}{b}+\frac{1}{c}\right)\right]}.$$

*Hint*: solve the problem $\Delta u_1 = 0$ for $a < r < c$, $\Delta u_2 = 0$ for $c < r < b$, $u_1(a) = u_0$, $u_2(b) = 0$, $u_1(c) = u_2(c)$, $\varepsilon_1 \frac{\partial u_1}{\partial n}\Big|_{r=c} = \varepsilon_2 \frac{\partial u}{\partial n}\Big|_{r=c}$, $u_0$ is the potential difference between the plates of the capacitor.

**39.** $$u = u_0 \frac{\frac{1}{r}+\frac{\varepsilon_1-\varepsilon_2}{\varepsilon_2}\frac{1}{c}}{\frac{1}{R}+\frac{\varepsilon_1-\varepsilon_2}{\varepsilon_2}\frac{1}{c}} \quad \text{for} \quad R < r < c,$$

$$u = u_0 \frac{\frac{\varepsilon_1}{\varepsilon_2}\frac{1}{r}}{\frac{1}{R}+\frac{\varepsilon_1-\varepsilon_2}{\varepsilon_2}\frac{1}{c}} \quad \text{for} \quad r > c.$$

**40.** (a) $$u = u_2+\frac{A}{4}(r^2-R_2^2)+\frac{u_1-u_2+0.25A(R_2^2-R_1^2)}{\ln R_2-\ln R_1}\ln\frac{R_2}{r};$$

(b) $$u = u_1+\frac{A}{4}(r^2-R_1^2)+R_2\left(u_2-\frac{A}{2}R_2\right)\ln\frac{R_2}{r}.$$

**41.** $$u = \frac{1}{6}(r^2-R_1^2)-\frac{1}{6}R_1R_2(R_1+R_2)\left(\frac{1}{R_1}-\frac{1}{r}\right).$$

**42.** $$u(x, y) = \frac{4V_0}{\pi}\sum_{n=0}^{\infty}(-1)^n \frac{\cosh\frac{2n+1}{2b}\pi x \cos\frac{2n+1}{2b}\pi y}{(2n+1)\cosh\frac{2n+1}{2b}\pi a}.$$

*Hint*: see Example 2 of Section 6.3.

CHAPTER 5

1. $u(x, t) = \frac{E_0}{2} e^{-x\sqrt{RG}}\left\{1-\Phi\left(x\sqrt{\frac{RC}{4t}}-\sqrt{\frac{Gt}{C}}\right)\right\}$

$$+\frac{E_0}{2} e^{x\sqrt{RG}}\left\{1-\Phi\left(x\sqrt{\frac{RC}{4t}}-\sqrt{\frac{Gt}{C}}\right)\right\}.$$

2. $G(r, r_0; t) = \frac{1}{8\pi r r_0\sqrt{\pi t}}\left[e^{-\frac{(r-r_0)^2}{4a^2t}} - e^{-\frac{(r+r_0)^2}{4a^2t}}\right].$

*Hint*: reduce the problem to the one-dimensional problem by substituting $u = v/r$: $a^2 v_{rr} = v_t$, $v(r, 0) = \frac{r}{4\pi}\frac{\delta(r-r_0)}{r_0^2}$.

3. $u(r, t) = \frac{u_0}{2}\left[\Phi\left(\frac{r+R}{\sqrt{4Dt}}\right) - \Phi\left(\frac{r-R}{\sqrt{4D}}\right.\right.$

$$+\frac{u_0}{r}\sqrt{\frac{Dt}{\pi}}\left[e^{-\frac{(r-R)^2}{4a^2t}} - e^{-\frac{(r+R)^2}{4a^2t}}\right].$$

4. (a) $u(x, y, z, t) = u\left(\sqrt{x^2+y^2+(z-z_0)^2}, t\right)$

$$+ u\left(\sqrt{x^2+y^2+(z+z_0)^2}, t\right),$$

(b) $u(x, y, z, t) = u\left(\sqrt{x^2+y^2+(z-z_0)^2}, t\right)$

$$- u\left(\sqrt{x^2+y^2+(z+z_0)^2}, t\right),$$

where $u(r, t)$ is the solution of the last problem.

5. $u(x, t) = \frac{1}{2}\int_{t_0}^{t}\left\{\Phi\left(\frac{b-x}{\sqrt{4a^2(t-\tau)}}\right)+\Phi\left(\frac{b+x}{\sqrt{4a^2(t-\tau)}}\right)\right.$

$$\left.-\Phi\left(\frac{a+x}{\sqrt{4a^2(t-\tau)}}\right)-\Phi\left(\frac{a-x}{\sqrt{4a^2(t-\tau)}}\right)\right\}Q(\tau)\,d\tau.$$

6. $u(r, t) = \frac{1}{\sqrt{4a^2\pi t}}\frac{1}{r}\int_0^\infty \xi^2\varphi(\xi)\left[e^{-\frac{(r-\xi)^2}{4a^2t}} - e^{-\frac{(r+\xi)^2}{4a^2t}}\right]d\xi$

$$+\frac{1}{r\sqrt{4\pi a^2}}\int_0^t\int_0^\infty \frac{\xi^2}{\sqrt{t-\tau}}f(\xi, \tau)\left[e^{-\frac{(r-\xi)^2}{4a^2(t-\tau)}} - e^{-\frac{(r+\xi)^2}{4a^2(t-\tau)}}\right]d\xi\,d\tau.$$

7. $$G(r, r_0; t) = \frac{1}{2\pi}\int_0^\infty e^{-a^2\lambda^2 t} J_0(\lambda r) J_0(\lambda r_0)\lambda \, d\lambda$$

$$= \frac{1}{4\pi a^2 t} e^{-\frac{r^2+r_0^2}{4a^2 t}} I_0\left(\frac{r r_0}{2a^2 t}\right).$$

*Hint*: solve the problem $a^2\left(u_{rr}+\frac{1}{r}u_r\right) = u_t$,

$$u(r, 0) = \frac{1}{2\pi}\frac{1}{r_0}\delta(r-r_0), \qquad |u| < \infty.$$

The solution must be sought in the form

$$u(r, t) = \int_0^\infty\int_0^\infty U(\rho, t) J_0(\lambda\rho) J_0(\lambda r)\lambda \, d\lambda \, d\rho \text{ (see Chapter 11).}$$

8. $$u(r, t) = \frac{1}{2a^2 t} e^{-\frac{r^2}{4a^2 t}} \int_0^\infty \xi\varphi(\xi) e^{-\frac{\xi^2}{4a^2 t}} I_0\left(\frac{r\xi}{2a^2 t}\right) d\xi$$

$$+\frac{1}{2a^2}\int_0^t\int_0^\infty \frac{\xi f(\xi, \tau)}{t-\tau} e^{-\frac{r^2+\xi^2}{4a^2(t-\tau)}} I_0\left(\frac{r\xi}{2a^2(t-\tau)}\right) d\xi \, d\tau.$$

9. $$G(x, x_0; t) = \frac{e^{-ht}}{\sqrt{4\pi a^2 t}} e^{-\frac{(x-x_0)^2}{4a^2 t}}.$$

## CHAPTER 6

1. **(a)** $$G(M, P) = \frac{1}{2\pi}\left(\ln\frac{1}{r_{MP}} - \ln\frac{R}{r_0 r_{MP_1}}\right),$$

where $P_1$ is the point symmetric to $P$ relative to the circle, $r_0$ is the distance of $P$ from the centre of the circle;

**(b)** $$G(M, P) = \frac{1}{4\pi}\left(\frac{1}{r_{MP}} - \frac{R}{r_0 r_{MP_1}}\right).$$

2. $$G(M, P) = \sum_{m=0}^{n-1} [G_{cr}(M, P_m) - G_{cr}(M, \bar{P}_m)],$$

where $G_{cr}(M, P)$ is Green's function for the interior of the circle, $P_m$ and $\overline{P}_m$ are points with the polar coordinates $\left(r_0, \varphi_0+\frac{2m}{n}\pi\right)$ and $\left(r_0, \frac{2m}{n}\pi-\varphi_0\right)$ respectively; $(r_0, \varphi_0)$ are the coordinates of $P$.

**3.** **(a)** $$G(M, P)=\sum_{n=0}^{\infty}\frac{1}{4\pi}\left(\frac{e_n}{r_{MP_n}}-\frac{e_n'}{r_{MP_n'}}\right),$$

where $P_n$ are the points with coordinates $(\rho_n, \theta_0, \varphi_0)$, $P_n'$ are the points with coordinates $(\varphi_n, \theta_0, \varphi_0)$ and $(\rho_0', \theta_0, \varphi_0)$ are the coordinates of $P$,

$$e_n=\begin{cases}\left(\dfrac{R_1}{R_2}\right)^k & \text{for } n=2k,\\[2ex] \left(\dfrac{R_2}{R_1}\right)^{k+1} & \text{for } n=2k+1,\end{cases}$$

$$e_n'=\begin{cases}\left(\dfrac{R_1}{R_2}\right)^k\dfrac{R_1}{\rho_0} & \text{for } n=2k,\\[2ex] \left(\dfrac{R_2}{R_1}\right)^k\dfrac{R_2}{\rho_0} & \text{for } n=2k+1,\end{cases}$$

$$\rho_n=\begin{cases}\left(\dfrac{R_1}{R_2}\right)^{2k}\rho_0 & \text{for } n=2k,\\[2ex] \left(\dfrac{R_2}{R_1}\right)^{2k+2}\rho_0 & \text{for } n=2k+1,\end{cases}$$

$$\rho_n'=\begin{cases}\left(\dfrac{R_1}{R_2}\right)^{2k}\dfrac{R_1^2}{\rho_0} & \text{for } n=2k,\\[2ex] \left(\dfrac{R_2}{R_1}\right)^{2k}\dfrac{R_2}{\rho_0} & \text{for } n=2k+1.\end{cases}$$

**(b)** $$G(M, P)=\frac{1}{2\pi}\sum_{n=0}^{\infty}\left(\ln\frac{e_n}{r_{MP_n}}-\ln\frac{e_n'}{r_{MP_n'}}\right)$$

where $P_n$ are points with coordinates $(\rho_n, \varphi_0)$, $P_n'$ are points with coordinates $(\rho_n', \varphi_0)$, $(\rho_0, \varphi_0)$ are the coordinates of $P$; $\rho_n, \rho_n'$, $e_n$ and $e_n'$ are determined by the same formulae as in case (a).

4. $$G(M, P) = \frac{1}{4\pi} \sum_{n=-\infty}^{\infty} \left\{ \frac{1}{r_{MP_n}} - \frac{1}{r_{MP'_n}} \right\},$$

where $P_n$ are points with coordinates $(x_0, y_0, z_0+2nh)$, $P'_n$ are points with coordinates $[x_0, y_0, z_0-(2n+1)h]$, $(x_0, y_0, z_0)$ are the coordinates of $P$.

## CHAPTER 7

1. $$u(r) = \begin{cases} B, & r \leqslant R_1, \\ -4\pi \int\limits_{R_1}^{r} (1/\xi - 1/r)\xi^2 \rho(\xi)\, d\xi + C, & R_1 \leqslant r \leqslant R_2, \\ D/r, & R_2 \leqslant r, \end{cases}$$

where $D = 4\pi \int\limits_{R_1}^{R_2} \xi^2 \rho(\xi)\, d\xi$,

$$B = C = D/R_2 + 4\pi \int\limits_{R_1}^{R_2} (1/\xi - 1/R_2)\xi^2 \rho(\xi)\, d\xi.$$

2. $$u(r) = \begin{cases} 4\pi R\rho_0, & r \leqslant R, \\ 4\pi R^2 \rho_0/r, & r \geqslant R. \end{cases}$$

3. $$u(r) = \begin{cases} 2\pi\rho_0(R^2 - r^2/3) - M/r_1, & r \leqslant R, \\ M(1/r - 1/r_1), & r \geqslant R, \end{cases}$$

where $M = 4/3\ \pi R^3 \rho_0$, $\quad r = \sqrt{x^2+y^2+(z-h)^2}$,

$r_1 = \sqrt{x^2+y^2+(z+h)^2}$,

$\rho_0$ is the charge density, $(0, 0, h)$ are the coordinates of the centre of the sphere of radius $R$.

*Hint*: calculate the effect of a perfectly conducting plane $z = 0$. Reflect the original sphere in the plane $z = 0$. The solution in this case then takes the form

$$u(r) = \begin{cases} C - \dfrac{2}{3}\pi\rho_0 r^2 - \dfrac{M}{r_1}, & r < R, \\ M\left(\dfrac{1}{r} - \dfrac{1}{r_1}\right), & r > R, \end{cases}$$

$C$ is determined from the condition that the solution must join at $r = R$.

4. $$u(r) = \begin{cases} M\left(\frac{1}{2} - \ln R - \frac{1}{2}\frac{r^2}{R^2}\right), & r \leqslant R, \\ M \ln \frac{1}{r}, & r \geqslant R, \end{cases}$$

where $M = \pi R^2 \rho_0$, $\rho_0$ is the charge density, $R$ is the radius of the circle.

5. The potential of the simple layer $0 \leqslant x < 1$ of density $\rho_0$ is

$$v(x, y) = \rho_0 \int_0^l \ln \frac{1}{\sqrt{(\xi - x)^2 + y^2}} \, d\xi .$$

6. $$w(M) = \nu_0 \int_0^l \frac{\cos \varphi}{r_{MP}} \, d\xi_P = \nu_0 y \int_0^l \frac{d\xi}{\sqrt{(\xi - x)^2 + y^2}} .$$

7. (a) $$w(\rho, \varphi) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(\rho^2 - R^2) f(\theta) \, d\theta}{R^2 + \rho^2 - 2R\rho \cos(\theta - \varphi)};$$

(b) $$w(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y f(\xi) \, d\xi}{(\xi - x)^2 + y^2} .$$

## CHAPTER 11

1. $$u(r, t) = 8A \sum_{n=1}^{\infty} \frac{1}{\alpha_n^3 J_1(\alpha_n)} e^{-\frac{a^2 \alpha_n^2}{R^2} t} J_0\left(\frac{\alpha_n}{R} r\right),$$

where $\alpha_n$ are the positive roots of the equation $J_0(\alpha) = 0$.

2. $$u(r, t) = \sum_{n=1}^{\infty} C_n e^{-\frac{a^2 \alpha_n^2}{R^2} t} J_0\left(\frac{\alpha_n}{R} r\right),$$

where $\alpha_n$ are the positive roots of the equation $\alpha J_0'(\alpha) + hR J_0(\alpha) = 0$,

$$C_n = \frac{1}{\left\| J_0\left(\frac{\alpha_n}{R} r\right) \right\|^2} \int_0^R f(\xi)\, \xi J_0\left(\frac{\alpha_n}{R}\xi\right) d\xi,$$

$f(r)$ is the bounded solution of the problem $\Delta f + \frac{JQ}{k} = 0$,

$f'(R)+hf(R) = 0$, $f(r) = \frac{Q_1}{6}(R^3 - r^3) + \frac{Q_1 R^2}{2h}$, $Q_1 = \frac{JQ}{k}$.

3. $$H(r, t) = H_0 - 2H_0 \sum_{n=1}^{\infty} \frac{1}{\alpha_n J_1(\alpha_n)} e^{-\frac{a^2\alpha_n^2 t}{R^2}} J_0\left(\frac{\alpha_n}{R} r\right),$$

where $\alpha_n$ are the positive roots of the equation $J_0(\alpha) = 0$. The flux of the magnetic induction through the cross-section of the cylinder is $\Phi = \int_0^R \int_0^{2\pi} \mu H(r, t) r\, dr\, d\varphi$, where $\mu$ is the permeability.

4. $$u(r, t) = f(r) + \sum_{n=1}^{\infty} C_n e^{-a^2\lambda_n t} \Phi_n(r),$$

where $\Phi_n(r) = J_0(\sqrt{\lambda_n} R_1) N_0(\sqrt{\lambda_n} r) - N_0(\sqrt{\lambda_n} R_1) J_0(\sqrt{\lambda_n} r)$,

$$C_n = \frac{1}{\|\Phi_n(r)\|^2} \int_{R_1}^{R_2} r f(r) \Phi_n(r)\, dr, \quad f(r) = \frac{qR_2}{k} \ln\frac{r}{R_1},$$

$\lambda_n$ are the positive roots of the equation $\Phi_n'(R_2) = 0$.

5. $a^2\Delta u + Q/\rho = u_{tt}$, $u(R, t) = 0$, $u(r, 0) = u_t(r, 0) = 0$, $|u| < \infty$,

$$u(r, t) = f(r) - \frac{2R^2 Q}{\rho a^2} \sum_{n=1}^{\infty} \frac{1}{\alpha_n^2 J_1(\alpha_n)} J_0\left(\frac{\alpha_n}{R} r\right) \cos\frac{a\alpha_n}{R} t,$$

where $\alpha_n$ are the positive roots of the equation $J_0(\alpha) = 0$,

$f(r) = \frac{R^2 - r^2}{4a^2\rho} Q$ is the stationary state.

6. (a) $$u(r, t) = f(r)\sin\omega t + \frac{2A\omega R^2}{a\rho} \sum_{n=1}^{\infty} \frac{J_0\left(\frac{\alpha_n}{R} r\right) \sin\frac{\alpha_n}{R} at}{\alpha_n^2(\omega^2 R^2 - a^2\alpha_n^2) J_1(\alpha_n)}$$

($\omega$ is not equal to any of the eigenvalues $a\alpha_n/R$),

$$f(r) = \frac{A}{\rho\omega^2}\left\{\frac{J_0\left(\frac{\omega}{a}r\right)}{J_0\left(\frac{\omega}{a}R\right)} - 1\right\}$$

**(b)** $u(r, t) = f(r)\cos\omega t + \frac{2AR}{\rho}\sum_{n=1}^{\infty}\frac{J_0\left(\frac{\alpha_n}{R}r\right)\cos\frac{\alpha_n a}{R}t}{\alpha_n(\omega^2R^2 - a^2\alpha_n^2)J_1(\alpha_n)}$;

**(c)** $u(r, t) = \sum_{n=1}^{\infty} J_0\left(\frac{\alpha_n}{R}r\right)\Psi_n(t)$,

where $\Psi_n(t) = B_n\int_0^t \sin\frac{a\alpha_n}{R}(t-\tau)\sin\omega\tau\, d\tau$,

$$B_n = \frac{2A}{a\alpha_n J_1^2(\alpha_n)}\left[R_2 J_1\left(\frac{\alpha_n}{R}R_2\right) - R_1 J_1\left(\frac{\alpha_n}{R}R_1\right)\right], \quad \text{for} \quad R_1 = R_2,$$

$$B_n = \frac{2A}{Ra\alpha_n}\frac{J_0\left(\frac{\alpha_n}{R}R_1\right)}{J_1^2(\alpha_n)}.$$

7. **(a)** $u(r, t) = f(r)\sin\omega t - \sum_{n=1}^{\infty} C_n J_0\left(\frac{\alpha_n r}{R}\right)\sin\frac{a\alpha_n}{R}t$,

where $C_n = \frac{2A\omega}{a\alpha_n R J_0\left(\frac{\omega}{a}R\right)J_1^2(\alpha_n)}\int_0^R rJ_0\left(\frac{\omega}{a}r\right)J_0\left(\frac{\alpha_n}{R}r\right)dr$,

$$f(r) = A\frac{J_0\left(\frac{\omega}{a}r\right)}{J_0\left(\frac{\omega}{a}R\right)},$$

$\alpha_n$ are the positive roots of the equation $J_0(\alpha) = 0$;

**(b)** $u(r, t) = f(r)\cos\omega t - \sum_{n=1}^{\infty} D_n J_0\left(\frac{\alpha_n}{R}r\right)\cos\frac{a\alpha_n}{R}t$,

where $D_n = C_n\frac{aR\alpha_n}{\omega}$.

**8.** $u(r, \varphi, t)$ is the solution of the problem $a^2\Delta u = u_{tt}$,

$$u(R, \varphi, t) = 0, \quad u(r, \varphi, 0) = 0, \quad u_t(r, \varphi, 0) = P\frac{1}{r}\delta(r-r_1)\delta(\varphi-\varphi_1),$$

$$u(r, \varphi, t) = \frac{2P}{\pi a R}\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{\cos n(\varphi-\varphi_1)J_n\left(\frac{\alpha_k^{(n)}}{R}r_1\right)}{\varepsilon_n\alpha_k^{(n)}[J_n'(\alpha_k^{(n)})]^2}J_n\left(\frac{\alpha_k^{(n)}}{R}r\right)\sin\frac{a\alpha_k^{(n)}}{R}t,$$

where $\varepsilon_n = \begin{cases} 1, & n \neq 0, \\ 2, & n = 0, \end{cases}$ $\alpha_k^{(n)}$ are the positive roots of the equations $J_n(\alpha) = 0$.

**9.** $$u(r, z) = V_0 - 2(V_0 - V_1)\sum_{n=1}^{\infty}\frac{\sinh\frac{\alpha_n}{R}z\, J_0\left(\frac{\alpha_n}{R}r\right)}{\sinh\frac{\alpha_n}{R}h\,\alpha_n J_1(\alpha_n)},$$

where $\alpha_n$ are the positive roots of the equation $J_0(\alpha) = 0$.

**10.** $$u(r, z) = \frac{Iz}{\pi R^2\sigma} + \frac{2I}{\pi R_1\sigma}\sum_{n=1}^{\infty}\frac{\sinh\frac{\alpha_n}{R}zJ_0\left(\frac{\alpha_n}{R}R_1\right)J_0\left(\frac{\alpha_n}{R}r\right)}{\cosh\frac{\alpha_n}{R}h\alpha_n^2J_0^2(\alpha_n)} + \text{const},$$

where $\alpha_n$ are the positive roots of the equation $J_1(\alpha) = 0$.

**11.** $$u(r, z) = \frac{Q}{2\pi k}\sum_{n=1}^{\infty}\frac{1}{\alpha_n^2 J_1^2(\alpha_n)}\left[1 - \frac{Rh_1\cosh\frac{\alpha_n}{R}z}{\alpha_n\sinh\frac{\alpha_n}{2R}h + Rh_1\cosh\frac{\alpha_n}{2R}h}\right]$$

$$\times J_0\left(\frac{\alpha_n}{R}r\right),$$

where $\alpha_n$ are the roots of the equation $J_0(\alpha) = 0$.

*Hint*: the solution $u(r, z)$ should be sought in the form $u = v(r) + w(r, z)$.

The formulation of the problems for $v(r)$ and $w(r, z)$ is:

$$v: \quad \Delta v + \frac{Q}{2\pi}\frac{\delta(r)}{r} = 0, \qquad |v| < \infty, \qquad v(R) = 0,$$

$$w: \quad \Delta w = 0, \qquad |w| < \infty, \qquad m(R, z) = 0,$$

$$w_z(r, -h/2) - h_1 w(r, -h/2) = h_1 v(r),$$

$$w_z(r, h/2) + h_1 w(r, h/2) = -h_1 v(r).$$

The origin of the coordinates should be taken in the centre of the cylinder.

**12.** $u(r,t) = F_0(r)+tF_1(r)-\sum_{n=1}^{\infty}\frac{2qR}{k}\frac{J_0\left(\frac{\alpha_n}{R}r\right)}{\alpha_n^2 J_0(\alpha_n)}e^{-\frac{a^2\alpha_n^2}{R^2}t}$,

where $F_0(r) = u_0-\frac{qR}{4k}\left(1-2\frac{r^2}{R^2}\right)$,

$F_1(r) = 2qa^2\frac{1}{kR}$, $\alpha_n$ are the positive roots of the equation $J_1(\alpha) = 0$.

**13.** **(a)** $\lambda_{n,m,k} = (\pi k/h)^2+\alpha_m^{(n)}/R)^2$, $\alpha_m^{(n)}$ are the positive roots of the equations $J_n(\alpha) = 0$,

$$\Phi_{n,m,k}(r, \varphi, z) = \sin\frac{\pi k}{h}zJ_n\left(\frac{\alpha_m^{(n)}}{R}r\right)\begin{cases}\sin n\varphi,\\ \cos n\varphi;\end{cases}$$

**(b)** $\lambda_{n,m,k} = (\pi k/h)^2+(\alpha_m^{(n)}/R)^2$, $\alpha_m^{(n)}$ are the positive roots of the equations $J_n'(\alpha) = 0$,

$$\Phi_{n,m,k}(r, \varphi, z) = \cos\frac{\pi k}{h}zJ_n\left(\frac{\alpha_m^{(n)}}{R}r\right)\begin{cases}\cos n\varphi,\\ \sin n\varphi;\end{cases}$$

**(c)** $\lambda_{n,m,k} = \nu_k^2+(\alpha_m^{(n)}/R)^2$, $\alpha_m^{(n)}$ are the positive roots of the equation $\alpha J_n'(\alpha)+RhJ_n(\alpha) = 0$, $\nu_k$ are the positive roots of the equation $\tan\nu h = \frac{(h_1+h_2)\nu}{\nu^2-h_1h_2}$.

$$\Phi_{n,m,k}(r, \varphi, z) = \Psi_k(z)J_n\left(\frac{\alpha_m^{(n)}}{R}r\right)\begin{cases}\cos n\varphi,\\ \sin n\varphi,\end{cases}$$

$\Psi_k(z) = \nu_k\cos\nu_k z+h_1\sin\nu_k z$.

**14.** **(a)** $\Phi_{n,m}(r,\varphi) = J_{n\pi/\alpha}\left(\frac{\gamma_m^{(n)}}{R}r\right)\sin\frac{\pi n}{\alpha}\varphi$, $\lambda_{n,m} = \left(\frac{\gamma_m^{(n)}}{R}\right)^2$,

$\gamma_m^{(n)}$ are the positive roots of the equations $J_{n\pi/\alpha}(\gamma) = 0$;

**(b)** $\Phi_{n,m}(r, \varphi) = J_{\pi n/\alpha}\left(\frac{\gamma_m^{(n)}}{R}r\right)\cos\frac{\pi n}{\alpha}\varphi$, $\lambda_{n,m} = \left(\frac{\gamma_m^{(n)}}{R}\right)^2$,

$\gamma_m^{(n)}$ are the positive roots of the equations $J_{n\pi/\alpha}'(\gamma) = 0$;

(c) $\Phi_{n,m}(r,\varphi) = J_{\nu_n}\left(\frac{\gamma_m^{(n)}}{R}r\right)\psi_n(\varphi), \quad \lambda_{n,m} = \left(\frac{\gamma_m^{(n)}}{R}\right)^2,$

$\gamma_m^{(n)}$ are the positive roots of the equations

$\gamma J'_{\nu_n}(\gamma)+RhJ_{\nu_n}(\gamma) = 0,$

$\nu_n$ are the positive roots of the equation

$$\tan\nu\alpha = \frac{(h_1+h_2)}{\nu^2-h_1h_2}\nu.$$

**15.** $$u(r,\varphi,t) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} C_{n,k}\,e^{-a^2\lambda_k^{(n)}t}\,J_{n\pi/\alpha}\left(\frac{\gamma_k^{(n)}}{R}r\right)\sin\frac{\pi n}{\alpha}\varphi,$$

$$C_{n,k} = \frac{4}{R^2\alpha[J'_{n\pi/\alpha}(\gamma_k^{(n)})]^2}\int_0^R\int_0^\alpha f(r,\varphi)rJ_{n\pi/\alpha}\left(\frac{\gamma_k^{(n)}}{R}r\right)\sin\frac{\pi n}{\alpha}\varphi\,dr\,d\varphi,$$

$\gamma_k^{(n)}$ are the positive roots of the equations $J_{n\pi/\alpha}(\gamma) = 0$, $\lambda_k^{(n)} = \left(\frac{\gamma_k^{(n)}}{R}\right)^2.$

**16.** $$u(r,\varphi,z,t) = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}(C_{n,m,k}\cos n\varphi$$

$$+D_{n,m,k}\sin n\varphi)J_n\left(\frac{\gamma_k^{(n)}}{R}r\right)\sin\frac{\pi m}{h}z\,e^{-a^2\lambda_{n,m,k}t},$$

where

$$C_{n,m,k} = \frac{4}{\varepsilon_n\pi R^2h[J'_n(\gamma_k^{(n)})]^2}\int_0^R\int_0^h\int_0^{2\pi} rf(r,\varphi,z)J_n\left(\frac{\gamma_k^{(n)}}{R}r\right)\sin\frac{\pi m}{h}z$$

$$\times\cos n\varphi\,dr\,dz\,d\varphi,$$

$$D_{n,m,k} = \frac{4}{\varepsilon_n\pi R^2h[J'_n(\gamma_k^{(n)})]^2}\int_0^R\int_0^h\int_0^{2\pi} rf(r,\varphi,z)J_n\left(\frac{\gamma_k^{(n)}}{R}r\right)\sin\frac{\pi m}{h}z$$

$$\times\sin n\varphi\,dr\,dz\,d\varphi,$$

$\gamma_k^{(n)}$ and $\lambda_{n,m,k}$ are determined as in Problem 13.

**19.** $$I_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sinh x, \quad I_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cosh x,$$

$$K_{\pm1/2}(x) = \sqrt{\frac{\pi}{2x}}e^{-x}, \quad N_{1/2}(x) = -J_{-1/2}(x) = -\sqrt{\frac{2}{\pi x}}\cos x,$$

$$N_{-1/2}(x) = J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x, \quad H^{(1)}_{1/2}(x) = -i\sqrt{\frac{2}{\pi x}}\,\mathrm{e}^{ix},$$

$$H^{(1)}_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\,\mathrm{e}^{ix}, \quad H^{(2)}_{1/2}(x) = -i\sqrt{\frac{2}{\pi x}}\,\mathrm{e}^{-ix},$$

$$H^{(2)}_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\,\mathrm{e}^{-ix}.$$

**20.** $u(r) = \dfrac{u_0}{I_0(\beta R)} I_0(\beta r), \quad \nabla^2 u - \beta^2 u = 0, \quad u|_{r=R} = u_0.$

**21.** $u(r) = \dfrac{u_0}{K_0(\beta R)} K_0(\beta r).$

**22.** $u(r, z) = \dfrac{u_2 - u_1}{h} z + u_1 + \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty} C_n I_0\left(\frac{\pi n}{h} r\right)\sin\frac{\pi n}{h} z,$

where $C_n = \dfrac{1}{n}\{(u_0 - u_1)[(-1)^n - 1] + (-1)^n (u_2 - u_1)\}\dfrac{1}{I_0\left(\dfrac{\pi n}{h} R\right)},$

$u(r, z)$ is the potential of the field $\boldsymbol{E}$, i.e. $\boldsymbol{E} = -\nabla^2 u$.

*Hint*: the solution should be sought in the form

$u = A(z) + B(r, z), \quad \nabla^2 A = 0, \quad A(0) = u_1, \quad A(h) = u_2.$

**23.** $u(r, z) = \displaystyle\sum_{n=1}^{\infty} C_n I_0\left(\frac{\alpha_n}{h} r\right)\cos\frac{\alpha_n}{h} z,$

where $C_n = \dfrac{1}{I_0\left(\dfrac{\alpha_n}{h} R\right)\left\|\cos\dfrac{\alpha_n}{h} z\right\|^2}\displaystyle\int_0^h f(z)\cos\frac{\alpha_n}{h} z\,dz.$

**24.** $u(r, z) = \dfrac{4u_0}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{K_0\left(\dfrac{\pi(2n-1)}{h} r\right)}{K_0\left(\dfrac{\pi(2n-1)}{h} R\right)}\sin\frac{\pi(2n-1)}{h} z.$

**25.** $u(r, z, t) = v(r, z) + \displaystyle\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} C_{n,k}\,\mathrm{e}^{-a^2\left(\alpha_n^2 + \frac{\pi^2 k^2}{h^2}\right)t}$

$$\times J_0\left(\frac{\alpha_n}{R} r\right)\sin\frac{\pi k}{h} z,$$

$$C_{n,k} = \frac{-2}{h J_1^2(\alpha_n)} \int_0^R \int_0^h v(r, z) r J_0\left(\frac{\alpha_n}{R} r\right) \sin\frac{\pi n}{h} z \, dz \, dr,$$

where $\alpha_n$ are the positive roots of the equation $J_0(\alpha) = 0$;

$$v(r, z) = \frac{u_0}{h} z + \frac{2u_0}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{I_0\left(\frac{\pi n}{h} R\right)} I_0\left(\frac{\pi n}{h} r\right) \sin\frac{\pi n}{h} z.$$

*Hint*: the solutions should be sought in the form
$u(r, z, t) = v(r, z) + w(r, z, t)$,
where $v(r, z)$ is the solution of the problem (see Problem 22):
$\nabla^2 v = 0, \quad v(R, z) = 0, \quad v(r, 0) = 0, \quad v(r, h) = u_0, \quad |v| < \infty.$

## CHAPTER 12

1. $P_{2k}(0) = (-1)^k \frac{(2k)!'}{2^{2k}(k!)^2}, \qquad P_{2k+1}(0) = 0.$

2. They are orthogonal with the weight $(1-x^2)^k$. This follows from the orthogonality of the associated Legendre functions.

3. $$u(x, t) = \sum_{n=1}^{\infty} \{C_n \cos a\sqrt{2n(2n-1)}\, t + D_n \sin a\sqrt{2n(2n-1)}\, t\} P_{2n-1}(x/l),$$

$$C_n = (4n-1) \int_0^l \varphi(\xi) P_{2n-1}\left(\frac{\xi}{l}\right) d\xi,$$

$$D_n = \frac{4n-1}{a\sqrt{2n(2n-1)}} \int_0^l \varphi_1(\xi) P_{2n-1}\left(\frac{\xi}{l}\right) d\xi.$$

4. $E = -\nabla u$, where $u(r, \theta)$ is the potential of the field

$$u(r, \theta) = \begin{cases} V_2 + \dfrac{V_1 - V_2}{2} \displaystyle\sum_{n=0}^{\infty} C_n \left(\frac{r}{R}\right)^{2n+1} P_{2n+1}(\cos\theta), & r \leqslant R, \\ V_2 + \dfrac{V_1 - V_2}{2} \displaystyle\sum_{n=0}^{\infty} C_n \left(\frac{R}{r}\right)^{2n+2} P_{2n+1}(\cos\theta), & r \geqslant R, \end{cases}$$

$$C_n = \frac{4n+3}{2n+2}\frac{(-1)^n(2n)!}{2^{2n}(n!)^2}.$$

5. (a) $u(r,\,\theta) = -e\sum_{n=0}^{\infty}\frac{r_0^n r^n}{R^{2n+1}}P_n(\cos\theta);$

(b) $u(r,\,\theta) = -e\sum_{n=0}^{\infty}\frac{R^{2n+1}}{r_0^{n+1}r^{n+1}}P_n(\cos\theta).$

*Hint*: the resultant potential $V(r,\,\theta)$, due to the point charge of the induced charges, should be sought in the form of the sum $V(r,\,\theta) = e/r_1+u(r,\,\theta)$,

$$\text{where } u(r,\,\theta) = \begin{cases} \sum_{n=0}^{\infty} C_n\left(\frac{r}{R}\right)^n P_n(\cos\theta), & r < R, \\ \sum_{n=0}^{\infty} D_n\left(\frac{R}{r}\right)^n P_n(\cos\theta), & r > R, \end{cases}$$

where $r_1$ is the distance between the point $(r,\,\theta)$ and the point $(r_0,\,0)$ at which the charge is located. Use the expansion

$$\frac{1}{r_1} = \begin{cases} \frac{1}{r_0}\sum_{n=0}^{\infty}\left(\frac{r}{r_0}\right)^n P_n(\cos\theta), & r < r_0, \\ \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r_0}{r}\right)^n P_n(\cos\theta), & r > r_0. \end{cases}$$

The coefficients $C_n$ and $D_n$ are found from the condition $V(R,\,\theta) = 0$:

$$C_n = -\frac{er_0^n}{R^{n+1}}, \qquad D_n = \frac{eR^n}{r_0^{n+1}}.$$

6. (a) If the charge lies outside the sphere at the point $(r_0,\,0)$, $r_0 > R$, the electrostatic potential is

$$u(r,\,\theta) = \begin{cases} u_1(r,\,\theta), & r \leqslant R, \\ u_2(r,\,\theta), & r \geqslant R, \end{cases}$$

where

$$u_1(r,\,\theta) = e\sum_{n=0}^{\infty}\frac{2n+1}{n\varepsilon_1+(n+1)\varepsilon_2}\,\frac{r^n}{r_0^{n+1}}P_n(\cos\theta),$$

$$u_2(r, \theta) = \frac{e}{\varepsilon_2 r_1} + e\frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2} \sum_{n=0}^{\infty} \frac{n}{n\varepsilon_1 + (n+1)\varepsilon_2} \frac{R^{2n+1}}{r_0^{n+1} r^{n+1}} P_n(\cos\theta).$$

*Hint*: the expansion coefficients are found from the conditions

$$u_1(R, \theta) = u_2(R, \theta), \qquad \varepsilon_1 \frac{\partial u_1}{\partial r}\bigg|_{r=R} = \varepsilon_2 \frac{\partial u_2}{\partial r}\bigg|_{r=R};$$

where $r_1$ is the distance from the point $(r, \theta)$ to the point $(r_0, 0)$, where the charge is located;

**(b)** If the charge lies outside the sphere ($r_0 < R$), then

$$u_1(r, \theta) = \frac{e}{\varepsilon_1 r_1} + \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1} e \sum_{n=0}^{\infty} \frac{n+1}{n\varepsilon_1 + (n+1)\varepsilon_2} \frac{r_0^n r^n}{R^{2n+1}} P_n(\cos\theta),$$

$$u_2(r, \theta) = e \sum_{n=0}^{\infty} \frac{2n+1}{n\varepsilon_1 + (n+1)\varepsilon_2} \frac{r_0^n}{r^{n+1}} P_n(\cos\theta).$$

7.

$$v(r, \theta) = \begin{cases} \dfrac{2e}{R} \displaystyle\sum_{n=0}^{\infty} \left(\frac{r}{R}\right)^n [P_n(0) + P_{n-2}(0)] P_n(\cos\theta) - \frac{2er}{R^2} P_1(\cos\theta), & r \leqslant R, \\ \dfrac{2e}{R} \displaystyle\sum_{n=0}^{\infty} \left(\frac{R}{r}\right)^{n+1} [P_n(0) + P_{n-2}(0)] P_n(\cos\theta), & r \geqslant R. \end{cases}$$

*Hint*: solve the problem by the method of separation of variables. Then

$$v(r, \theta) = \begin{cases} \displaystyle\sum_{n=0}^{\infty} C_n r^n P_n(\cos\theta), & r \leqslant R, \\ \displaystyle\sum_{n=0}^{\infty} D_n \frac{1}{r^{n+1}} P_n(\cos\theta), & r \geqslant R. \end{cases}$$

The coefficients $C_n$ and $D_n$ are found by comparing these formulae with the expansion in powers of $z$ for the potential at a point on the $z$ axis (which is perpendicular to the disc and passes through its centre) which is calculated directly:

$$V(z, 0) = \frac{2e}{R}\left\{\sqrt{z^2 + R^2} - z\right\}.$$

**8.** $$v(r, \theta) = \frac{I}{2\pi\sigma R} \sum_{n=0}^{\infty} \frac{4n+3}{2n+1} \left(\frac{r}{R}\right)^{2n+1} P_{2n+1}(\cos\theta).$$

In view of the symmetry of the problem $v(r, \pi/2) = 0$. Therefore, in the expansion $v(r, \theta) = \sum\limits_{n=0}^{\infty} C_k r^k P_k(\cos\theta)$ the coefficients $C_{2n}$ with even indices should vanish. The remaining coefficients are determined from $-\sigma v_r(R, \theta) = \dfrac{1}{2\pi R^2} \dfrac{\delta(\theta)}{\sin\theta} I$.

**9.** $$u(r, \theta) = \frac{Q}{4\pi k r_1} + \frac{Q}{4\pi k} \sum_{n=0}^{\infty} \frac{n+1-Rh}{n+Rh} \frac{r_0^n r^n}{R^{2n+1}} P_n(\cos\theta),$$

where $r_1$ is the distance between the point $(r, \theta)$ and the source $(r_0, 0)$.

*Hint*: the solutions should be sought in the form of the sum

$$u(r, \theta) = \frac{Q}{4\pi k r_1} + v(r, \theta),$$

where $v(r, \theta)$ is the solution of the problem

$$\Delta v = 0, \qquad k v_r(R, \theta) + h v(R, \theta) = \frac{-Q}{4\pi k} \left\{ k \frac{\partial}{\partial r}\left(\frac{1}{r_1}\right) + \frac{h}{r_1} \right\}_{r=R}.$$

Use the expansion of $1/r_1$ into a series in terms of the Legendre polynomials.

**10.** $$u(r, \theta) = \frac{e}{r_1} - e \sum_{n=0}^{\infty} \left\{ \frac{r_0^{2n+1} - R_1^{2n+1}}{R_2^{2n+1} - R_1^{2n+1}} \frac{r^n}{r_0^{n+1}} + \frac{R_2^{2n+1} - r_0^{2n+1}}{R_2^{2n+1} - R_1^{2n+1}} \frac{R_1^{2n+1}}{(r_0 r)^{n+1}} \right\} P_n(\cos\theta).$$

The density of the induced charges is

$$\sigma_1 = \sigma|_{r=R_1} = \frac{-e}{4\pi} \sum_{n=0}^{\infty} \frac{(2n+1)(R_2^{2n+1} - r_0^{2n+1})}{R_2^{2n+1} - R_1^{2n+1}} \frac{R_1^{n-1}}{r_0^{n+1}} P_n(\cos\theta),$$

$$\sigma_2 = \sigma|_{r=R_2} = \frac{e}{4\pi} \sum_{n=0}^{\infty} \frac{(2n+1)(r_0^{2n+1} - R_1^{2n+1})}{R_2^{2n+1} - R_1^{2n+1}} \frac{R_2^{n-1}}{r_0^{n+1}} P_n(\cos\theta),$$

where $r_1$ is the distance between the point $(r, \theta)$ and the charge placed at $(r_0, 0)$.

*Hint*: the solution should be sought in the form

$$u = e/r_1 + v(r, \theta), \qquad \sigma_i = -\frac{1}{4\pi}\left(\frac{\partial u}{\partial n}\right)_{r=R_i} \qquad (i = 1, 2).$$

**11.** $$u(r, \theta) = \frac{u_0}{2}\left\{1 - \cos\alpha - \sum_{n=1}^{\infty} [P_{n+1}(\cos\alpha) - P_{n-1}(\cos\alpha)]\left(\frac{r}{R}\right)^n P_n(\cos\theta)\right\}.$$

**12.** $$u(r, \theta) = \frac{qR}{2k}\left\{\frac{1}{2hR} + \frac{r}{R}\frac{\cos\theta}{1+Rh} - \sum_{n=1}^{\infty} \frac{(4n+1)P_{2n}(0)r^{2n}P_{2n}(\cos\theta)}{(2n+hR)(2n-1)(2n+2)R^{2n}}\right\}.$$

*Hint*: the boundary condition for the problem is

$$u_r(R, \theta) + hu(R, \theta) = \begin{cases} q/k \cos\theta, & 0 \leqslant \theta \leqslant \pi/2 \\ 0, & \pi/2 \leqslant \theta \leqslant \pi. \end{cases}$$

**13.** $$u(r, \theta, t) = \frac{r^n}{nR^{n-1}} f(t) P_n(\cos\theta) + P_n(\cos\theta)\sum_{k=1}^{\infty} \psi_k(t)\frac{1}{\sqrt{r}} J_{n+1/2}\left(\frac{\alpha_k}{R} r\right),$$

where $\alpha_k$ are the positive roots of the equation

$$\alpha J_{n+1/2}(\alpha) - \frac{1}{2} J_{n+1/2}(\alpha) = 0,$$

$$\psi_k(t) = \frac{RA_k}{\alpha_k a}\int_0^t f''(\tau)\sin\frac{a\alpha_k}{R}(t-\tau)\,d\tau,$$

$$A_k = \frac{-2}{nR^{n+1}} \frac{\int_0^R r^{n+3/2} J_{n+1/2}\left(\frac{\alpha_k}{R} r\right) dr}{J_{n+1/2}^2(\alpha_k)\left[1 - \frac{n(n+1)}{\alpha_k^2}\right]}.$$

where $u(r, \theta, t)$ is the velocity potential $a^2\nabla^2 u = u_{tt}$, $u(r, \theta, 0) = u_t(r, \theta, 0) = 0$, $u_r(R, \theta, t) = P_n(\cos\theta)f(t)$, $|u| < \infty$.

**14.** $\lambda_{n,m,k} = 1/R^2(\alpha_m^{(n)})^2 + k^2$ are the eigenvalues, and

$$\Phi_{n,m,k}(r,\theta,\varphi)=\frac{1}{\sqrt{r}}J_{n+1/2}\left(\frac{\alpha_m^{(n)}}{R}r\right)P_n^k(\cos\theta)\begin{cases}\cos k\varphi,\\ \sin k\varphi\end{cases}$$

are the eigenfunctions, where $\alpha_m^{(n)}$ are the positive roots of the following equations:

**(a)** $J_{n+1/2}(\alpha)=0,$

**(b)** $J'_{n+1/2}(\alpha)-\dfrac{1}{2\alpha}J_{n+1/2}(\alpha)=0,$

**(c)** $2\alpha J'_{n+1/2}(\alpha)-(1-2Rh)J_{n+1/2}(\alpha)=0,$

$h$ is the constant in the condition $v_r(R,\theta,\varphi)+hv(R,\theta,\varphi)=0.$

**15.** $$u(r,\theta,\varphi)=\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\{C_{n,m,k}\cos k\varphi+D_{n,m,k}\sin k\varphi\}$$

$$\times\frac{1}{\sqrt{r}}J_{n+1/2}\left(\frac{\alpha_m^{(n)}}{R}r\right)P_n^k(\cos\theta)\,e^{-\frac{a^2[\alpha_m^{(n)}]^2}{R^2}t},$$

where $\alpha_m^{(n)}$ are the positive roots of the equation $J_{n+1/2}(\alpha)=0,$

$$C_{n,m,k}=A_{n,m,k}\int_0^R\int_0^\pi\int_0^{2\pi}f(r,\theta,\varphi)r^{3/2}J_{n+1/2}\left(\frac{\alpha_m^{(n)}}{R}r\right)P_n^k(\cos\theta)\cos k\varphi$$
$$\times\sin\theta\,d\varphi\,d\theta\,dr,$$

$$D_{n,m,k}=A_{n,m,k}\int_0^R\int_0^\pi\int_0^{2\pi}f(r,\theta,\varphi)r^{3/2}J_{n+1/2}\left(\frac{\alpha_m^{(n)}}{R}r\right)P_n^k(\cos\theta)\sin k\varphi$$
$$\times\sin\theta\,d\varphi\,d\theta\,dr,$$

$$A_{n,m,k}=\frac{(2n+1)(n-k)!}{\pi R^2(n+k)!\,\varepsilon_k J_{n+1/2}^2(\alpha_m^{(n)})},\qquad \varepsilon_k=\begin{cases}2, & k=0,\\ 1, & k\neq 0.\end{cases}$$

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